Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

The table shows the unit costs of transporting goods from supply points to demand points. In each case:

a use the north-west corner method to find initial solution,

b verify that, for each solution, the number of occupied cells = number of supply points + number of demand points − 1.

c determine the cost of each initial solution.

	P	Q	R	Supply
A	150	213	222	32
B	175	204	218	44
C	188	198	246	34
Demand	28	45	37	

Solution:

a

	P	Q	R	Supply
A	28	4		32
B		41	3	44
C			34	34
Demand	28	45	37	110

b Supply points = 3, demand points = 3, occupied cells = 5.
$3+3-1=5=$ number of occupied cells. Yes formula holds.

	P	Q	R	Supply
A	150	213	222	32
B	175	204	218	44
C	188	198	246	34
Demand	28	45	37	110

c Cost $= 28\times150+4\times213+41\times204+3\times218+34\times246 = 22\,434$

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Exercise A, Question 2

Question:

The table shows the unit costs of transporting goods from supply points to demand points. In each case:

a use the north-west corner method to find initial solution,

b verify that, for each solution, the number of occupied cells $=$ number of supply points $+$ number of demand points -1.

c determine the cost of each initial solution.

	P	**Q**	**R**	**S**	**Supply**
A	27	33	34	41	54
B	31	29	37	30	67
C	40	32	28	35	29
Demand	21	32	51	46	

Solution:

a

	P	Q	R	S	Supply
A	21	32	1		54
B			50	17	67
C				29	29
Demand	21	32	51	46	150

b Supply points $= 3$, demand points $= 4$, occupied cells $= 6$.
$3 + 4 - 1 = 6 =$ number of occupied cells. Yes formula holds.

	P	Q	R	S	Supply
A	27	33	34	41	54
B	31	29	37	30	67
C	40	32	28	35	29
Demand	21	32	51	46	150

c Cost $= 21 \times 27 + 32 \times 33 + 1 \times 34 + 50 \times 37 + 17 \times 30 + 29 \times 35 = 5032$

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Exercise A, Question 3

Question:

The table shows the unit costs of transporting goods from supply points to demand points. In each case:

a use the north-west corner method to find initial solution,

b verify that, for each solution, the number of occupied cells = number of supply points + number of demand points − 1.

c determine the cost of each initial solution.

	P	Q	R	Supply
A	17	24	19	123
B	15	21	25	143
C	19	22	18	84
D	20	27	16	150
Demand	200	100	200	500

Solution:

a

	P	Q	R	Supply
A	123			123
B	77	66		143
C		34	50	84
D			150	150
Demand	200	100	200	500

b Supply points = 4, demand points = 3. Occupied cells = 6.

$4+3-1=6=$ number of occupied cells. Yes formula holds.

	P	Q	R	Supply
A	17	24	19	123
B	15	21	25	143
C	19	22	18	84
D	20	27	16	150
Demand	200	100	200	500

c Cost $=123\times17+77\times15+66\times21+34\times22+50\times18+150\times16=8680$

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Exercise A, Question 4

Question:

The table shows the unit costs of transporting goods from supply points to demand points. In each case:

a use the north-west corner method to find initial solution,

b verify that, for each solution, the number of occupied cells = number of supply points + number of demand points − 1.

c determine the cost of each initial solution.

	P	Q	R	S	Supply
A	56	86	80	61	134
B	59	76	78	65	203
C	62	70	57	67	176
D	60	68	75	71	187
Demand	175	175	175	175	700

Solution:

a

	P	Q	R	S	Supply
A	134				134
B	41	162			203
C		13	163		176
D			12	175	187
Demand	175	175	175	175	700

b Supply points $= 4$, demand points $= 4$, occupied cells $= 7$.
$4+4-1=7=$ number of occupied cells. Yes formula holds.

	P	Q	R	S	Supply
A	56	86	80	61	134
B	59	76	78	65	203
C	62	70	57	67	176
D	60	68	75	71	187
Demand	175	175	175	175	700

c $\text{Cost} = 134\times56+41\times59+162\times76+13\times70+163\times57+12\times75+175\times71$
$= 45\,761$

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Exercise A, Question 5

Question:

	A	B	C	D	Supply
X	27	33	34	41	60
Y	31	29	37	30	60
Z	40	32	28	35	80
Demand	40	70	50	20	

Four sandwich shops A, B, C and D can be supplied with bread from three bakeries, X, Y, and Z. The table shows the cost, in pence, of transporting one tray of bread from each supplier to each shop, the number of trays of bread required by each shop and the number of trays of bread that can be supplied by each bakery.

a Explain why it is necessary to add a dummy demand point in order to solve this problem, and what this dummy point means in practical terms.

b Use the north-west corner method to determine an initial solution to this problem and the cost of this solution.

Solution:

a The total supply is 200, but the total demand is 180. A dummy is needed to absorb this excess, so that total supply equals total demand.

b

	A	B	C	D	Supply
X	27	33	34	41	60
Y	31	29	37	30	60
Z	40	32	28	35	80
Demand	40	70	50	20	

Becomes

	A	B	C	D	Dummy	Supply
X	27	33	34	41	0	60
Y	31	29	37	30	0	60
Z	40	32	28	35	0	80
Demand	40	70	50	20	20	200

North-west corner solution is

	A	B	C	D	Dummy	Supply
X	40	20				60
Y		50	10			60
Z			40	20	20	80
Demand	40	70	50	20	20	200

$$\text{Cost} = 40 \times 27 + 20 \times 33 + 50 \times 29 + 10 \times 37 + 40 \times 28 + 20 \times 35 + 20 \times 0 = 5\,380$$

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Exercise A, Question 6

Question:

	K	L	M	N	Supply
A	35	46	62	80	20
B	24	53	73	52	15
C	67	61	50	65	20
D	92	81	41	42	20
Demand	25	10	18	22	

A company needs to supply ready-mixed concrete from four depots A, B, C and D to four work sites K, L, M and N. The number of loads that can be supplied from each depot and the number of loads required at each site are shown in the table above, as well as the transportation cost per load from each depot to each work site.

a Explain what is meant by a degenerate solution.

b Demonstrate that the north-west corner method gives a degenerate solution.

c Adapt your solution to give a non-degenerate initial solution.

Solution:

a A degenerate solution occurs when the number of cells used in a solution is fewer than the number of rows + number of columns − 1. It will happen when an entry, other than the last, completes both the supply requirement of the row and the demand requirement of the column.

b

	K	L	M	N	Stock
A	20				20
B	5	10			15
C			18	2	20
D				20	20
Demand	25	10	18	22	

c

	K	L	M	N	Stock
A	20				20
B	5	10			15
C		0	18	2	20
D				20	20
Demand	25	10	18	22	

or

	K	L	M	N	Stock
A	20				20
B	5	10	0		15
C			18	2	20
D				20	20
Demand	25	10	18	22	

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Exercise A, Question 7

Question:

	L	**M**	**N**	**Supply**
P	3	5	9	22
Q	4	3	7	a
R	6	4	8	11
S	8	2	5	b
Demand	15	17	20	

The table shows a balanced transportation problem. The initial solution, given by the north-west corner method, is degenerate.

a Use this information to determine the values of a and b.

b Hence write down the initial, degenerate solution given by the north west-corner method.

Solution:

	L	M	N	Supply
P	3	5	9	22
Q	4	3	7	a
R	6	4	8	11
S	8	2	5	b
Demand	15	17	20	

a Since the problem is **balanced** the total supply = the total demand, giving

$a+b+22+11=15+17+20$

hence $a+b=\mathbf{19}$ (1)

Since the initial north-west corner solution is **degenerate** we know that the supply and demand are both met before the final entry.

(Since $22+a+11>15+17$ we know that this must occur before row 3)

Hence $22+a=15+17$, giving $a=10$

Using equation (1) we get $b=9$

So the values are: $a=10$ and $b=9$

b

	L	M	N	Supply
P	15	7		22
Q		10		10
R			11	11
S			9	9
Demand	15	17	20	

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Exercise B, Question 1

Question:

Start with the initial, north-west corner, solutions found in question 1 of exercise 1A.
In each case use the initial solution, and the original cost matrix, shown below,
to find

a the shadow costs,

b the improvement indices

c the entering cell, if appropriate.

	P	Q	R	Supply
A	150	213	222	32
B	175	204	218	44
C	188	198	246	34
Demand	28	45	37	

Solution:

a

Shadow costs		150	213	227	
		P	Q	R	Supply
0	A	150	213	222	32
−9	B	175	204	218	44
19	C	188	198	246	34
	Demand	28	45	37	110

b Improvement indices for cells:

$BP = 175 + 9 - 150 = 34$

$CP = 188 - 19 - 150 = 19$

$CQ = 198 - 19 - 213 = -34$

$AR = 222 - 0 - 227 = -5$

c Entering cell is CQ, since it has the most negative improvement index.

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Exercise B, Question 2

Question:

Start with the initial, north-west corner, solutions found in question 2 of exercise 1A.
In each case use the initial solution, and the original cost matrix, shown below, to find

a the shadow costs,
b the improvement indices
c the entering cell, if appropriate.

	P	Q	R	S	Supply
A	27	33	34	41	54
B	31	29	37	30	67
C	40	32	28	35	29
Demand	21	32	51	46	

Solution:

a

Shadow costs		27	33	34	27	
		P	Q	R	S	Supply
0	A	27	33	34	41	54
3	B	31	29	37	30	67
8	C	40	32	28	35	29
	Demand	21	32	51	46	

b Improvement indices for cells:

$BP = 31 - 3 - 27 = 1$

$CP = 40 - 8 - 27 = 5$

$BQ = 29 - 3 - 33 = 7$

$CQ = 32 - 8 - 33 = -9$

$CR = 28 - 8 - 34 = -14$

$AS = 41 - 0 - 27 = 14$

c Entering cell is CR, since it has the most negative improvement index.

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Exercise B, Question 3

Question:

Start with the initial, north-west corner, solutions found in question 3 of exercise 1A.
In each case use the initial solution, and the original cost matrix, shown below, to find

a the shadow costs,
b the improvement indices
c the entering cell, if appropriate.

	P	**Q**	**R**	**Supply**
A	17	24	19	123
B	15	21	25	143
C	19	22	18	84
D	20	27	16	150
Demand	200	100	200	

Solution:

a

Shadow costs		17	23	19	
		P	Q	R	Supply
0	A	17	24	19	123
−2	B	15	21	25	143
−1	C	19	22	18	84
−3	D	20	27	16	150
	Demand	200	100	200	

b Improvement indices for cells:

$CP = 19+1-17=3$
$DP = 20+3-17=6$
$AQ = 24-0-23=1$
$DQ = 27+3-23=7$
$AR = 19-0-19=0$
$BR = 25+2-19=8$

c There are no negative improvement indices, so the solution is optimal.

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Exercise B, Question 4

Question:

Start with the initial, north-west corner, solutions found in question 4 of exercise 1A.
In each case use the initial solution, and the original cost matrix, shown below,
to find

a the shadow costs,
b the improvement indices
c the entering cell, if appropriate.

	P	Q	R	S	Supply
A	56	86	80	61	134
B	59	76	78	65	203
C	62	70	57	67	176
D	60	68	75	71	187
Demand	175	175	175	175	

Solution:

a

Shadow costs		56	73	60	56	
		P	Q	R	S	Supply
0	A	56	86	80	61	134
3	B	59	76	78	65	203
-3	C	62	70	57	67	176
15	D	60	68	75	71	187
	Demand	175	175	175	175	

b Improvement indices for cells:

$$\text{CP} = 62+3-56=9$$
$$\text{DP} = 60-15-56=-11$$
$$\text{AQ} = 86-0-73=13$$
$$\text{DQ} = 68-15-73=-20$$
$$\text{AR} = 80-0-60=20$$
$$\text{BR} = 78-3-60=15$$
$$\text{AS} = 61-0-56=5$$
$$\text{BS} = 65-3-56=6$$
$$\text{CS} = 67+3-56=14$$

c Entering cell is DQ, since it has the most negative improvement index.

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Exercise C, Question 1

Question:

Complete your solutions to the transportation problems from question 1 in exercise 1A. You should demonstrate that your solution is optimal.

	P	Q	R	Supply
A	150	213	222	32
B	175	204	218	44
C	188	198	246	34
Demand	28	45	37	

Solution:

	P	Q	R	Supply
A	150	213	222	32
B	175	204	218	44
C	188	198	246	34
Demand	28	45	37	110

Our current solution is

	P	Q	R	Supply
A	28	4		32
B		41	3	44
C			34	34
Demand	28	45	37	110

We established in question 1 of exercise 1B that the entering cell was CQ, so we enter θ into cell CQ and get the following stepping stone route

	P	Q	R	Supply
A	28	4		32
B		$41-\theta$	$3+\theta$	44
C		θ	$34-\theta$	34
Demand	28	45	37	110

The largest possible value of θ is 34, making CR the exiting cell, and giving the improved solution.

	P	Q	R	Supply
A	28	4		32
B		7	37	44
C		34		34
Demand	28	45	37	110

This gives the following shadow costs:

Shadow costs		150	213	227	
		P	Q	R	Supply
0	A	150	213	222	32
−9	B	175	204	218	44
−15	C	188	198	246	34
	Demand	28	45	37	110

Improvement indices for cells:

$BP = 175 + 9 - 150 = 34$

$CP = 188 + 15 - 150 = 53$

$AR = 222 - 0 - 227 = -5$

$CR = 246 + 15 - 227 = 34$

The solution is not optimal, since we have a negative improvement index and the new entering cell is AR

We insert θ into cell AR and set the following stepping stone route

	P	Q	R	Supply
A	28	$4-\theta$	θ	32
B		$7+\theta$	$37-\theta$	44
C		34		34
Demand	28	45	37	50

The maximum value of θ is 4 making AQ the exiting cell
Improved solution

	P	Q	R	Supply
A	28		4	32
B		11	33	44
C		34		34
Demand	28	45	37	150

Shadow costs

		150	208	222	
		P	Q	R	Supply
0	A	150	213	222	32
−4	B	175	204	218	44
−10	C	188	198	246	34
	Demand	28	45	37	150

Improvement indices

$AQ = 213-0-208=5$
$BP = 175+4-150=29$
$CP = 188+10-150=48$
$CR = 246+10-222=34$

Since all improvement indices are non-negative we have the optimal solution

	P	Q	R	Supply
A	28		4	32
B		11	33	44
C		34		34
Demand	28	45	37	150

28 units A to P
4 units A to R
11 units B to Q
33 units B to R
34 units C to Q cost 21 258

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Exercise C, Question 2

Question:

Complete your solutions to the transportation problems from question 2 in exercise 1A. You should demonstrate that your solution is optimal.

	P	Q	R	S	Supply
A	27	33	34	41	54
B	31	29	37	30	67
C	40	32	28	35	29
Demand	21	32	51	46	

Solution:

	P	Q	R	S	Supply
A	27	33	34	41	54
B	31	29	37	30	67
C	40	32	28	35	29
Demand	21	32	51	46	150

Our current solution from question 2 of exercise 1A is

	P	Q	R	S	Supply
A	21	32	1		54
B			50	17	67
C				29	29
Demand	21	32	51	46	150

We established in question 2 of exercise 1B that the entering cell was CR, so we enter θ into cell CR and get the following stepping stone route

	P	Q	R	S	Supply
A	21	32	1		54
B			$50-\theta$	$17+\theta$	67
C			θ	$29-\theta$	29
Demand	21	32	51	46	150

The largest possible value of θ is 29, making CS the exiting cell and giving the improved solution

	P	Q	R	S	Supply
A	21	32	1		54
B			21	46	67
C			29		29
Demand	21	32	51	46	150

This gives the following shadow costs:

Shadow costs		27	33	34	27	
		P	Q	R	S	Supply
0	A	27	33	34	41	54
3	B	31	29	37	30	67
−6	C	40	32	28	35	29
	Demand	21	32	51	46	150

Improvement indices

$AS = 41 - 0 - 27 = 14$
$BP = 31 - 3 - 27 = 1$
$BQ = 29 - 3 - 33 = -7$
$CP = 40 + 6 - 27 = 19$
$CQ = 32 + 6 - 33 = 5$
$CS = 35 + 6 - 27 = 14$

The entering cell must be BQ, giving the following stepping stone route

	P	Q	R	S	Supply
A	21	$32-\theta$	$1+\theta$		54
B		θ	$21-\theta$	46	67
C			29		29
Demand	21	32	51	46	150

The largest possible value of θ is 21, making BR the exiting cell and giving the following improved solution.

	P	Q	R	S	Supply
A	21	11	22		54
B		21		46	67
C			29		29
Demand	21	32	51	46	150

This gives the following shadow costs

Shadow costs		27	33	34	34	
		P	Q	R	S	Supply
0	A	27	33	34	41	54
-4	B	31	29	37	30	67
-6	C	40	32	28	35	29
	Demand	21	32	51	46	150

Improvement indices:

$$AS = 41 - 0 - 34 = 7$$
$$BP = 31 + 4 - 27 = 8$$
$$BR = 37 + 4 - 34 = 7$$
$$CP = 40 + 6 - 27 = 19$$
$$CQ = 32 + 6 - 33 = 5$$
$$CS = 35 + 6 - 34 = 7$$

All improvement indices are non-negative so our solution is optimal

Optimal solution
21 units A to P
11 units A to Q
22 units A to R
21 units B to Q
46 units B to S
29 units C to R Cost 4479

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Exercise C, Question 3

Question:

Complete your solutions to the transportation problems from question 4 in exercise 1B. You should demonstrate that your solution is optimal.

	P	Q	R	S	Supply
A	56	86	80	61	134
B	59	76	78	65	203
C	62	70	57	67	176
D	60	68	75	71	187
Demand	175	175	175	175	

The solution to question **3** requires a number of iterations, plus the optimality check – you will certainly get lots of practise in implementing the algorithms!

Solution:

	P	Q	R	S	Supply
A	56	86	80	61	134
B	59	76	78	65	203
C	62	70	57	67	176
D	60	68	75	71	187
Demand	175	175	175	175	700

Initial solution (from question 4 of exercise 1A)

	P	Q	R	S	Supply
A	134				134
B	41	162			203
C		13	163		176
D			12	175	187
Demand	175	175	175	175	700

The entering cell is DQ (from question 4 of exercise 1B)

Stepping stone route

	P	Q	R	S	Supply
A	134				134
B	41	162			203
C		$13-\theta$	$163+\theta$		176
D		θ	$12-\theta$	175	187
Demand	175	175	175	175	700

$\theta = 12$ Exiting cell is CS
improved solution

	P	Q	R	S	Supply
A	134				134
B	41	162			203
C		1	175		176
D		12		175	187
Demand	175	175	175	175	700

Shadow costs		56	73	60	76	
		P	Q	R	S	Supply
0	A	56	86	80	61	134
3	B	59	76	78	65	203
−3	C	62	70	57	67	176
−5	D	60	68	75	71	187
	Demand	175	175	175	175	700

Improvement indices: $AQ = 86 - 0 - 73 = 13$
$AR = 80 - 0 - 60 = 20$
$AS = 61 - 0 - 76 = -15$
$BR = 78 - 3 - 60 = 15$
$BS = 65 - 3 - 76 = -14$
$CP = 62 + 3 - 56 = 9$
$CS = 67 + 3 - 76 = -6$
$DP = 60 + 5 - 56 = 9$
$DR = 75 + 5 - 60 = 20$

Entering cell AS
stepping stone route

	P	Q	R	S	Supply
A	$134-\theta$			θ	134
B	$41+\theta$	$162-\theta$			203
C		1	175		176
D		$12+\theta$		$175-\theta$	187
Demand	175	175	175	175	700

Greatest value of θ is 134
Exiting cell is AP.

	P	Q	R	S	Supply
A				134	134
B	175	28			203
C		1	175		176
D		146		41	187
Demand	175	175	175	175	700

Shadow costs		41	58	45	61	
		P	Q	R	S	Supply
0	A	56	86	80	61	134
18	B	59	76	78	65	203
12	C	62	70	57	67	176
10	D	60	68	75	71	187
	Demand	175	175	175	175	700

Improvement indices: $AP = 56 - 0 - 41 = 15$

$AQ = 86 - 0 - 58 = 28$

$AR = 80 - 0 - 45 = 35$

$BR = 78 - 18 - 45 = 15$

$BS = 65 - 18 - 61 = -14$

$CP = 62 - 12 - 41 = 9$

$CS = 67 - 12 - 61 = -6$

$DP = 60 - 10 - 41 = 9$

$DR = 75 - 10 - 45 = 20$

Entering cell is BS

	P	Q	R	S	Supply
A				134	134
B	175	$28-\theta$		θ	203
C		1	175		176
D		$146+\theta$		$41-\theta$	187
Demand	175	175	175	175	700

$\theta = 28$ Exiting cell is BQ

	P	Q	R	S	Supply
A				134	134
B	175			28	203
C		1	175		176
D		174		13	187
Demand	175	175	175	175	700

Shadow costs		55	58	45	61	
		P	Q	R	S	Supply
0	A	56	86	80	61	134
4	B	59	76	78	65	203
12	C	62	70	57	67	176
10	D	60	68	75	71	187
	Demand	175	175	175	175	700

Entering cell is CS

	P	Q	R	S	Supply
A				134	134
B	175			28	203
C		$1-\theta$	175	θ	176
D		$174+\theta$		$13-\theta$	187
Demand	175	175	175	175	700

$\theta = 1$ Exiting cell CQ

	P	Q	R	S	Supply
A				134	134
B	175			28	203
C			175	1	176
D		175		12	187
Demand	175	175	175	175	700

Shadow costs		55	58	51	61	
		P	Q	R	S	Supply
0	A	56	86	80	61	134
4	B	59	76	78	65	203
6	C	62	70	57	67	176
10	D	60	68	75	71	187
	Demand	175	175	175	175	700

Improvement indices:

$$
\begin{aligned}
AP &= 56-0-55=1\\
AQ &= 86-0-58=28\\
AR &= 80-0-51=29\\
BQ &= 76-4-58=14\\
BR &= 78-4-51=23\\
CP &= 62-6-55=1\\
CQ &= 70-6-58=6\\
DP &= 60-10-55=-5\\
DR &= 75-10-51=14
\end{aligned}
$$

Entering cell is DP.

	P	Q	R	S	Supply
A				134	134
B	$175-\theta$			$28+\theta$	203
C			175	1	176
D	θ	175		$12-\theta$	187
Demand	175	175	175	175	700

$\theta = 12$ Exiting cell is DS

	P	Q	R	S	Supply
A				134	134
B	163			40	203
C			175	1	176
D	12	175			187
Demand	175	175	175	175	700

Shadow costs		55	63	51	61	
		P	Q	R	S	Supply
0	A	56	86	80	61	134
4	B	59	76	78	65	203
6	C	62	70	57	67	176
5	D	60	68	75	71	187
	Demand	175	175	175	175	700

Improvement indices: $AP = 56-0-55=1$

$AQ = 86-0-63=23$

$AR = 80-0-51=29$

$BQ = 76-4-63=9$

$BR = 78-4-51=23$

$CP = 62-6-55=1$

$CQ = 70-6-63=1$

$DR = 75-5-51=19$

$DS = 71-5-61=5$

No negative improvement indices so our solution is optimal.

134 units A to S
163 units B to P
40 units B to S
175 units C to R
1 unit C to S
12 units D to P
175 units D to Q Cost 43 053

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Exercise C, Question 4

Question:

	P	Q	Stock
A	2	6	3
B	2	7	5
C	6	9	2
Demand	6	4	

The table shows the unit cost, in pounds, of transporting goods from each of three warehouses A, B and C to each of two supermarkets P and Q. It also shows the stock at each warehouse and the demand at each supermarket.
Solve the transportation problem shown in the table. Use the north-west corner method to obtain an initial solution. You must state your shadow costs, improvement indices, stepping-stone routes, θ values, entering cells and exiting cells. You must state the initial cost and the improved cost after each iteration.

Solution:

	P	Q	Stock
A	2	6	3
B	2	7	5
C	6	9	2
Demand	6	4	10

Initial solution

	P	Q	Stock
A	3		3
B	3	2	5
C		2	2
Demand	6	4	10

Shadow costs		2	7	
		P	Q	Stock
0	A	2	6	3
0	B	2	7	5
2	C	6	9	2
	Demand	6	4	10

Improvement indices

$AQ = 6 - 0 - 7 = -1$

$CP = 6 - 2 - 2 = 2$

AQ is the entering cell

	P	Q	Stock
A	$3-\theta$	θ	3
B	$3+\theta$	$2-\theta$	5
C		2	2
Demand	6	4	10

$\theta = 2$ Exiting cell BQ.

	P	Q	Stock
A	1	2	3
B	5		5
C		2	2
Demand	6	4	

Shadow costs		2	6	
		P	Q	Stock
0	A	2	6	3
0	B	2	7	5
3	C	6	9	2
	Demand	6	4	

Improvement indices

$BQ = 7 - 0 - 6 = 1$
$CP = 6 - 3 - 2 = 1$

There are no negative improvement indices so our solution is optimal

1 unit from A to P
2 units from A to Q
5 units from B to P
2 units from C to Q Cost 42

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Exercise D, Question 1

Question:

Formulate the following transportation problem as a linear programming problem.

	P	Q	R	Supply
A	150	213	222	32
B	175	204	218	44
C	188	198	246	34
Demand	28	45	37	

Solution:

Let x_{ij} be the number of units transported from i to j where

$i \in \{A, B, C\}$

$j \in \{P, Q, R\}$

$x_{ij} \geq 0$

$$\text{Minimise } C = 150x_{11} + 213x_{12} + 222x_{13} + 175x_{21} + 204x_{22} + 218x_{23} + 188x_{31} + 198x_{32} + 246x_{33}$$

Subject to

$$\begin{aligned} x_{11} + x_{12} + x_{13} &\leq 32 \\ x_{21} + x_{22} + x_{23} &\leq 44 \\ x_{31} + x_{32} + x_{33} &\leq 34 \\ x_{11} + x_{21} + x_{31} &\leq 28 \\ x_{12} + x_{22} + x_{32} &\leq 45 \\ x_{13} + x_{23} + x_{33} &\leq 37 \end{aligned}$$

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Exercise D, Question 2

Question:

Formulate the following transportation problem as a linear programming problem.

	P	Q	R	S	Supply
A	27	33	34	41	54
B	31	29	37	30	67
C	40	32	28	35	29
Demand	21	32	51	46	

Solution:

Let x_{ij} be the number of units transported from i to j where

$i \in \{A, B, C\}$

$j \in \{P, Q, R, S\}$

$x_{ij} \geq 0$

$$\begin{aligned}\text{Minimise } C = {} & 27x_{11} + 33x_{12} + 34x_{13} + 41x_{14} \\ & 31x_{21} + 29x_{22} + 37x_{23} + 30x_{24} \\ & 40x_{31} + 32x_{32} + 28x_{33} + 35x_{34}\end{aligned}$$

$$\begin{aligned}\text{Subject to } & x_{11} + x_{12} + x_{13} + x_{14} \leq 54 \\ & x_{21} + x_{22} + x_{23} + x_{24} \leq 67 \\ & x_{31} + x_{32} + x_{33} + x_{34} \leq 29 \\ & x_{11} + x_{21} + x_{31} \leq 21 \\ & x_{12} + x_{22} + x_{32} \leq 32 \\ & x_{13} + x_{23} + x_{33} \leq 51 \\ & x_{14} + x_{24} + x_{34} \leq 46\end{aligned}$$

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise D, Question 3

Question:

Formulate the following transportation problem as a linear programming problem.

	P	Q	R	Supply
A	17	24	19	123
B	15	21	25	143
C	19	22	18	84
D	20	27	16	150
Demand	200	100	200	

Solution:

Let x_{ij} be the number of units transported from i to j where $i \in \{A, B, C, D\}$
and $j \in \{P, Q, R\}$
$x_{ij} \geq 0$

$$\begin{aligned} \text{Minimise } C = {} & 17x_{11} + 24x_{12} + 19x_{13} \\ & + 15x_{21} + 21x_{22} + 25x_{23} \\ & + 19x_{31} + 22x_{32} + 18x_{33} \\ & + 20x_{41} + 27x_{42} + 16x_{43} \end{aligned}$$

$$\begin{aligned} \text{Subject to} \quad x_{11} + x_{12} + x_{13} &\leq 123 \\ x_{21} + x_{22} + x_{23} &\leq 143 \\ x_{31} + x_{32} + x_{33} &\leq 84 \\ x_{41} + x_{42} + x_{43} &\leq 150 \\ x_{11} + x_{21} + x_{31} + x_{41} &\leq 200 \\ x_{12} + x_{22} + x_{32} + x_{42} &\leq 100 \\ x_{13} + x_{23} + x_{33} + x_{43} &\leq 200 \end{aligned}$$

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Exercise D, Question 4

Question:

Formulate the following transportation problem as a linear programming problem.

	P	Q	R	S	Supply
A	56	86	80	61	134
B	59	76	78	65	203
C	62	70	57	67	176
D	60	68	75	71	187
Demand	175	175	175	175	

Solution:

Let x_{ij} be the number of units transported from i to j where $i \in \{A, B, C, D\}$ and $j \in \{P, Q, R, S\}$

$$\begin{aligned}\text{Minimise } C = {} & 56x_{11} + 86x_{12} + 80x_{13} + 61x_{14} \\ & + 59x_{21} + 76x_{22} + 78x_{23} + 65x_{24} \\ & + 62x_{31} + 70x_{32} + 57x_{33} + 67x_{34} \\ & + 60x_{41} + 68x_{42} + 75x_{43} + 71x_{44}\end{aligned}$$

$$\begin{aligned}\text{Subject to } & x_{11} + x_{12} + x_{13} + x_{14} \le 134 \\ & x_{21} + x_{22} + x_{23} + x_{24} \le 203 \\ & x_{31} + x_{32} + x_{33} + x_{34} \le 176 \\ & x_{41} + x_{42} + x_{43} + x_{44} \le 187 \\ & x_{11} + x_{21} + x_{31} + x_{41} \le 175 \\ & x_{12} + x_{22} + x_{32} + x_{42} \le 175 \\ & x_{13} + x_{23} + x_{33} + x_{43} \le 175 \\ & x_{14} + x_{24} + x_{34} + x_{44} \le 175\end{aligned}$$

Solutionbank D2

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Exercise E, Question 1

Question:

	L	M	Supply
A	20	70	15
B	40	30	5
C	60	90	8
Demand	16	12	

The table shows the cost, in pounds, of transporting a car from each of three factories A, B and C to each of two showrooms L and M. It also shows the number of cars available for delivery at each factory and the number required at each showroom.

a Use the north-west corner method to find an initial solution.

b Solve the transportation problem, stating shadow costs, improvement indices, entering cells, stepping-stone routes, θ values and exiting cells.

c Demonstrate that your solution is optimal and find the cost of your optimal solution.

d Formulate this problem as a linear programming problem, making your decision variables, objective function and constraints clear.

e Verify that your optimal solution lies in the feasible region of the linear programming problem.

Solution:

a

	L	M	Stock
A	15		15
B	1	4	5
C		8	8
Demand	16	12	28

b

Shadow costs		20	10	
		L	M	Stock
0	A	20	70	15
20	B	40	30	5
80	C	60	90	8
		16	12	28

Improvement indices.

$AM = 70 - 0 - 10 = 60$

$CL = 60 - 80 - 20 = -40$

Entering cell CL $\theta = 1$

Exiting cell BL

	L	M	Stock
A	15		15
B	$1-\theta$	$4+\theta$	5
C	θ	$8-\theta$	8
Demand	16	12	28

	L	M	Stock
A	15		15
B		5	5
C	1	7	8
Demand	16	12	28

Shadow costs		20	50	
		L	M	Stock
0	A	20	70	15
−20	B	40	30	5
40	C	60	90	8
	Demand	16	12	28

c Improvement indices:

$AM = 70 - 0 - 50 = 20$

$BL = 40 + 20 - 20 = 40$

No negative improvement indices, so optimal solution.

15 units from A to L

5 units from B to M Cost: 1 140

1 unit from C to L

7 units from C to M

d Let x_{ij} be the number of units transported from i to j where $i \in \{A, B, C\}$ and $j \in \{L, M\}$

$x_{ij} \geq 0$

$$\text{Minimise } C = 20x_{11} + 70x_{12} + 40x_{21} + 30x_{22} + 60x_{31} + 90x_{32}$$

Subject to

$$x_{11} + x_{12} \leq 15$$
$$x_{21} + x_{22} \leq 5$$
$$x_{31} + x_{32} \leq 8$$
$$x_{11} + x_{21} + x_{31} \leq 16$$
$$x_{12} + x_{22} + x_{32} \leq 12$$

e In our solution

$x_{11} = 15 \quad x_{12} = 0$

$x_{21} = 0 \quad x_{22} = 5$

$x_{31} = 1 \quad x_{32} = 7$

$$x_{11} + x_{12} \leq 15 \quad 15 + 0 \leq 15$$
$$x_{21} + x_{22} \leq 5 \quad 0 + 5 \leq 5$$
$$x_{31} + x_{32} \leq 8 \quad 1 + 7 \leq 8$$
$$x_{11} + x_{21} + x_{31} \leq 16 \quad 15 + 0 + 1 \leq 16$$
$$x_{12} + x_{22} + x_{32} \leq 12 \quad 0 + 5 + 7 \leq 12$$

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Exercise E, Question 2

Question:

	P	Q	R	Supply
F	23	21	22	15
G	21	23	24	35
H	22	21	23	10
Demand	10	30	20	

The table shows the cost of transporting one unit of stock from each of three supply points F, G and H to each of three sales points P, Q and R. It also shows the stock held at each supply point and the amount required at each sales point.

a Use the north-west corner method to obtain an initial solution.
b Taking the most negative improvement index to indicate the entering square, perform two complete iterations of the stepping-stone method. You must state your shadow costs, improvement indices, stepping-stone routes and exiting cells.
c Explain how you can tell that your current solution is optimal.
d State the cost of your optimal solution.
e Taking the zero improvement index to indicate the entering square, perform one further iteration to obtain a second optimal solution.

Solution:

a

	P	Q	R	Stock
F	10	5		15
G		25	10	35
H			10	10
Demand	10	30	20	60

b

Shadow cost		23	21	22	
		P	Q	R	Stock
0	F	23	21	22	15
2	G	21	23	24	35
1	H	22	21	23	10
	Demand	10	30	20	60

Improvement indices:

$FR = 22 - 0 - 22 = 0$

$GP = 21 - 2 - 23 = -4$

$HP = 22 - 1 - 23 = -2$

$HQ = 21 - 1 - 21 = -1$

Entering cell is GP.

	P	Q	R	Stock
F	$10-\theta$	$5+\theta$		15
G	θ	$25-\theta$	10	35
H			10	10
Demand	10	30	20	60

$\theta = 10$ exiting cell FP
improved solution:

	P	Q	R	Stock
F		15		15
G	10	15	10	35
H			10	10
Demand	10	30	20	60

Shadow cost		19	21	22	
		P	Q	R	Stock
0	F	23	21	22	15
2	G	21	23	24	35
1	H	22	21	23	10
	Demand	10	30	20	60

Improvement indices:

$FP = 23 - 0 - 19 = 4$

$FR = 22 - 0 - 22 = 0$

$HP = 22 - 1 - 19 = 2$

$HQ = 21 - 1 - 21 = -1$

Entering cell is HQ

	P	Q	R	Stock
F		15		15
G	10	$15-\theta$	$10+\theta$	35
H		θ	$10-\theta$	10
Demand	10	30	20	60

$\theta = 10$

Exiting cell is HR
Improved solution

	P	Q	R	Stock
F		15		15
G	10	5	20	35
H		10		10
Demand	10	30	20	60

Shadow costs		19	21	22	
		P	Q	R	Stock
0	F	23	21	22	15
2	G	21	23	24	35
0	H	22	21	23	10
	Demand	10	30	20	60

Improvement indices:

FP $= 23 - 0 - 19 = 4$

FR $= 22 - 0 - 22 = 0$

HP $= 22 - 0 - 19 = 3$

HR $= 23 - 0 - 22 = 1$

c No negative improvement indices, so optimal solution

d 15 units F to Q

10 units G to P
5 units G to Q
20 units G to R
10 units H to Q

Cost: 1 330

e entering cell FR.

	P	Q	R	Stock
F		$15-\theta$	θ	15
G	10	$5+\theta$	$20-\theta$	35
H		10		10
Demand	10	30	20	60

$\theta = 15$

exiting cell FQ

second solution

	P	Q	R	Stock
F			15	15
G	10	20	5	35
H		10		10
Demand	10	30	20	60

Cost also 1330

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Exercise E, Question 3

Question:

	X	Y	Z	Supply
J	8	5	7	30
K	5	5	9	40
L	7	2	10	50
M	6	3	15	50
Demand	25	45	100	

The transportation problem represented by the table above is to be solved. A possible north-west corner solution is

	X	Y	Z	Supply
J	25	5		30
K		40		40
L		0	50	50
M			50	50
Demand	25	45	100	

a Explain why it is was necessary to add a zero entry (in cell LY) to the solution.
b State the cost of this initial solution.
c Choosing cell MX as the entering cell, perform one iteration of the stepping-stone method to obtain an improved solution. You must make your route clear, state your exiting cell and the cost of the improved solution.
d Determine whether your current solution is optimal. Give a reason for your answer.

After two more iterations the following solution was found.

	X	Y	Z	Supply
J			30	30
K		20	20	40
L			50	50
M	25	25		50
Demand	25	45	100	

e Taking the most negative improvement index to indicate the entering square, perform one further complete iteration of the stepping-stone method. You must state your shadow costs, improvement indices, stepping-stone route and exiting cell.

Solution:

a Otherwise the solution would be degenerate.

b 1675

c

	X	Y	Z	Supply
J	$25-\theta$	$5+\theta$		30
K		40		40
L		$0-\theta$	$50+\theta$	50
M	θ		$50-\theta$	50
Demand	25	45	100	170

The exiting cell is LY $\theta = 0$

	X	Y	Z	Supply
J	25	5		30
K		40		40
L			50	50
M	0		50	50
Demand	25	45	100	170

The cost is unchanged at 1675

d

Shadow costs		8	5	17	
		X	Y	Z	Supply
0	J	8	5	7	30
0	K	5	5	9	40
-7	L	7	2	10	50
-2	M	6	3	15	50
	Demand	25	45	100	170

Improvement indices:

JZ $= 7-0-17 = -10$

KX $= 5-0-8 = -3$

KZ $= 9-0-17 = -8$

LX $= 7+7-8 = 6$

LY $= 2+7-5 = 4$

MY $= 3+2-5 = 0$

This solution is not optimal since there are negative improvement indices.

e

	X	Y	Z	Supply
J			30	30
K		20	20	40
L			50	50
M	25	25		50
Demand	25	45	100	170

Shadow costs		6	3	7	
		X	Y	Z	Supply
0	J	8	5	7	30
2	K	5	5	9	40
3	L	7	2	10	50
0	M	6	3	15	50
	Demand	25	45	100	170

Improvement indices:

$$\begin{aligned} JX &= 8-0-6=2 \\ JY &= 5-0-3=2 \\ KX &= 5-2-6=-3 \\ LX &= 7-3-6=-2 \\ LY &= 2-3-3=-4 \\ MZ &= 15-0-7=8 \end{aligned}$$

Entering square is LY

	X	Y	Z	Supply
J			30	30
K		$20-\theta$	$20+\theta$	40
L		θ	$50-\theta$	50
M	25	25		50
Demand	25	45	100	170

$\theta = 20$ Exiting square is KY.
Improved solution

	X	Y	Z	Supply
J			30	30
K			40	40
L		20	30	50
M	25	25		50
Demand	25	45	100	170

Shadow costs		2	−1	7	
		X	Y	Z	Supply
0	J	8	5	7	30
2	K	5	5	9	40
3	L	7	2	10	50
4	M	6	3	15	50
	Demand	25	45	100	170

Improvement indices:

$JX = 8 - 0 - 2 = 6$

$JY = 5 - 0 + 1 = 6$

$KX = 5 - 2 - 2 = 1$

$KY = 5 - 2 + 1 = 4$

$LX = 7 - 3 - 2 = 2$

$MZ = 15 - 4 - 7 = 4$

All improvement indices are non-negative, so we have an optimal solution.

25 units M to X
25 units M to Y
20 units L to Y — Cost 1135
30 units L to Z
40 units K to Z
30 units J to Z

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Exercise E, Question 4

Question:

	S	T	U	Supply
A	6	10	7	50
B	7	5	8	70
C	6	7	7	50
Demand	100	30	20	

a Explain why a dummy demand point might be needed when solving a transportation problem.

The table shows the cost, in pounds, of transporting one van load of fruit tree seedlings from each of three greenhouses A, B and C to three garden centres S, T and U. It also shows the stock held at each greenhouse and the amount required at each garden centre.

b Use the north-west corner method to obtain an initial solution.

c Taking the most negative improvement index in each case to indicate the entering square, use the stepping-stone method to obtain an optimal solution. You must state your shadow costs, improvement indices, stepping-stone routes, entering squares and exiting cells.

d State the cost of your optimal solution.

e Formulate this problem as a linear programming problem. Make your decision variables, objective function and constraints clear.

Solution:

a The total demand is 150, the total stock is 170 so demand $<$ stock
We need a dummy demand to absorb the surplus stock.

	S	T	U	Dummy	Stock
A	6	10	7	0	50
B	7	5	8	0	70
C	6	7	7	0	50
Demand	100	30	20	20	170

b Initial solution

	S	T	U	Dummy	Stock
A	50				50
B	50	20			70
C		10	20	20	50
Demand	100	30	20	20	170

c

Shadow costs		6	4	4	−3	
		S	T	U	Dummy	Stock
0	A	6	10	7	0	50
1	B	7	5	8	0	70
3	C	6	7	7	0	50
	Demand	100	30	20	20	170

Improvement indices:

$$\begin{aligned} AT &= 10-0-4=6 \\ AU &= 7-0-4=3 \\ A\ \text{Dummy} &= 0-0+3=3 \\ BU &= 8-1-4=3 \\ B\ \text{Dummy} &= 0-1+3=2 \\ CS &= 6-3-6=-3 \end{aligned}$$

	S	T	U	Dummy	Stock
A	50				50
B	$50-\theta$	$20+\theta$			70
C	θ	$10-\theta$	20	20	50
Demand	100	30	20	20	170

Entering square CS $\theta=10$
Exiting square CT

Improved solution

	S	T	U	Dummy	Stock
A	50				50
B	40	30			70
C	10		20	20	50
Demand	100	30	20	20	170

Shadow costs		6	4	7	0	
		S	T	U	Dummy	Stock
0	A	6	10	7	0	50
1	B	7	5	8	0	70
0	C	6	7	7	0	50
	Demand	100	30	20	20	170

Improvement indices:

$$\begin{aligned} AT &= 10-0-4=6 \\ AU &= 7-0-7=0 \\ A\text{ Dummy} &= 0-0+0=0 \\ BU &= 8-1-7=0 \\ B\text{ Dummy} &= 0-1-0=-1 \\ CT &= 7-0-4=3 \end{aligned}$$

	S	T	U	Dummy	Stock
A	50				50
B	$40-\theta$	30		θ	70
C	$10+\theta$		20	$20-\theta$	50
Demand	100	30	20	20	170

Entering square B Dummy
$\theta = 20$

Exiting square C Dummy.

Improved solution

	S	T	U	Dummy	Stock
A	50				50
B	20	30		20	70
C	30		20		50
Demand	100	30	20	20	170

Shadow costs		6	4	7	−1	
		S	T	U	Dummy	Stock
0	A	6	10	7	0	50
1	B	7	5	8	0	70
0	C	6	7	7	0	50
	Demand	100	30	20	20	170

Improvement indices:

$$\begin{aligned}
\text{AT} &= 10-0-4=6\\
\text{AU} &= 7-0-7=0\\
\text{A Dummy} &= 0-0+1=1\\
\text{BU} &= 8-1-7=0\\
\text{CT} &= 7-0-4=3\\
\text{C Dummy} &= 0-0+1=1
\end{aligned}$$

The two zero improvement indices indicate that there are two further optimal solutions.

Using AU as an entering square, we get

	S	T	U	Dummy
A	$50-\theta$		θ	
B	20	30		20
C	$30+\theta$		$20-\theta$	

→

	S	T	U	Dummy	Stock
A	30		20		50
B	20	30		20	70
C	50				50
Demand	100	30	20	20	170

Using BU as an entering square, we get

	S	T	U	Dummy
A	50			
B	$20-\theta$	30	θ	20
C	$30+\theta$		$20-\theta$	

→

	S	T	U	Dummy	Stock
A	50				50
B		30	20	20	70
C	50		0		50
Demand	100	30	20	20	170

	S	T	U	Dummy	Stock
A	50				50
B	0	30	20	20	70
C	50				50
Demand	100	30	20	20	170

d Cost 910

e Let x_{ij} be the number of units transported from i to j where $i \in \{A, B, C\}$ and $j \in \{S, T, U, \text{dummy}\}$

$x_{ij} \geq 0$

$$\begin{aligned} \text{Minimize } C = {} & 6x_{11} + 10x_{12} + 7x_{13} \\ & + 7x_{21} + 5x_{22} + 8x_{23} \\ & + 6x_{31} + 7x_{32} + 7x_{33} \end{aligned}$$

$$\begin{aligned} \text{Subject to } & x_{11} + x_{12} + x_{13} + x_{14} \leq 50 \\ & x_{21} + x_{22} + x_{23} + x_{24} \leq 70 \\ & x_{31} + x_{32} + x_{33} + x_{34} \leq 50 \\ & x_{11} + x_{21} + x_{31} \leq 100 \\ & x_{12} + x_{22} + x_{32} \leq 30 \\ & x_{13} + x_{23} + x_{33} \leq 20 \\ & x_{14} + x_{24} + x_{34} \leq 20 \end{aligned}$$

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

The table shows the cost, in pounds, of allocating workers to tasks.
Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, showing the table at each stage and state your final solution and its cost.

	Task X	**Task Y**	**Task Z**
Worker A	34	35	31
Worker B	26	31	27
Worker C	30	37	32

Solution:

$$\begin{pmatrix} 34 & 35 & 31 \\ 26 & 31 & 27 \\ 30 & 37 & 32 \end{pmatrix} \rightarrow \begin{matrix}\text{reducing} \\ \text{rows}\end{matrix} \rightarrow \begin{pmatrix} 3 & 4 & 0 \\ 0 & 5 & 1 \\ 0 & 7 & 2 \end{pmatrix} \rightarrow$$

$$\begin{matrix}\text{reducing} \\ \text{columns}\end{matrix} \rightarrow \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{pmatrix}$$

$$\text{Minimum uncovered is } 1 \quad \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

Two solutions:

A – Y (35)		A – Z (31)	
B – Z (27)	or	B – Y (31)	cost 92
C – X (30)		C – X (30)	

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Exercise A, Question 2

Question:

The table shows the cost, in pounds, of allocating workers to tasks.
Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, showing the table at each stage and state your final solution and its cost.

	Task A	Task B	Task C	Task D
Worker P	34	37	32	32
Worker Q	35	32	34	37
Worker R	42	35	37	36
Worker S	38	34	35	39

Solution:

$$\begin{pmatrix} 34 & 37 & 32 & 32 \\ 35 & 32 & 34 & 37 \\ 42 & 35 & 37 & 36 \\ 38 & 34 & 35 & 39 \end{pmatrix} \rightarrow \begin{matrix}\text{reducing} \\ \text{rows}\end{matrix} \rightarrow \begin{pmatrix} 2 & 5 & 0 & 0 \\ 3 & 0 & 2 & 5 \\ 7 & 0 & 2 & 1 \\ 4 & 0 & 1 & 5 \end{pmatrix}$$

$$\rightarrow \begin{matrix}\text{reducing} \\ \text{columns}\end{matrix} \begin{pmatrix} 0 & 5 & 0 & 0 \\ 1 & 0 & 2 & 5 \\ 5 & 0 & 2 & 1 \\ 2 & 0 & 1 & 5 \end{pmatrix}$$

$$\text{Minimum uncovered is 1} \begin{pmatrix} 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 4 & 0 & 1 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}$$

Two solutions

$$\begin{matrix} P-A\ (34) \\ Q-B\ (32) \\ R-D\ (36) \\ S-C\ (35) \end{matrix} \quad \text{or} \quad \begin{matrix} P-D\ (32) \\ Q-A\ (35) \\ R-B\ (35) \\ S-C\ (35) \end{matrix} \quad \text{cost } 137$$

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Edexcel AS and A Level Modular Mathematics

Exercise A, Question 3

Question:

The table shows the cost, in pounds, of allocating workers to tasks.
Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, showing the table at each stage and state your final solution and its cost.

	Task R	Task S	Task T	Task U
Worker J	20	22	14	24
Worker K	20	19	12	20
Worker L	13	10	18	16
Worker M	22	23	9	28

Solution:

$$\begin{pmatrix} 20 & 22 & 14 & 24 \\ 20 & 19 & 12 & 20 \\ 13 & 10 & 18 & 16 \\ 22 & 23 & 9 & 28 \end{pmatrix} \rightarrow \begin{matrix}\text{reducing} \\ \text{rows}\end{matrix} \rightarrow \begin{pmatrix} 6 & 8 & 0 & 10 \\ 8 & 7 & 0 & 8 \\ 3 & 0 & 8 & 6 \\ 13 & 14 & 0 & 19 \end{pmatrix}$$

$$\begin{matrix}\text{reducing} \\ \text{columns}\end{matrix} \begin{pmatrix} 3 & 8 & 0 & 4 \\ 5 & 7 & 0 & 2 \\ 0 & 0 & 8 & 0 \\ 10 & 14 & 0 & 13 \end{pmatrix}$$

Minimum uncovered element is 2 $\begin{pmatrix} 1 & 6 & 0 & 2 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 8 & 12 & 0 & 11 \end{pmatrix}$

Minimum uncovered element is 1 $\begin{pmatrix} 0 & 5 & 0 & 1 \\ 3 & 5 & 1 & 0 \\ 0 & 0 & 11 & 0 \\ 7 & 11 & 0 & 10 \end{pmatrix}$ or Minimum uncovered element is 1 $\begin{pmatrix} 0 & 5 & 0 & 2 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 11 & 0 \\ 7 & 11 & 0 & 10 \end{pmatrix}$

Solution

J – R (20)
K – U (20)
L – S (10)
M – T (9)

cost 59

Solutionbank D2

Edexcel AS and A Level Modular Mathematics

Exercise A, Question 4

Question:

The table shows the cost, in pounds, of allocating workers to tasks.
Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, showing the table at each stage and state your final solution and its cost.

	Task V	Task W	Task X	Task Y	Task Z
Worker D	85	95	97	87	80
Worker E	110	115	95	105	100
Worker F	90	95	86	93	105
Worker G	85	83	84	85	87
Worker H	100	100	105	120	95

Solution:

$$\begin{pmatrix} 85 & 95 & 97 & 87 & 80 \\ 110 & 115 & 95 & 105 & 100 \\ 90 & 95 & 86 & 93 & 105 \\ 85 & 83 & 84 & 85 & 87 \\ 100 & 100 & 105 & 120 & 95 \end{pmatrix} \rightarrow \begin{matrix}\text{reducing}\\ \text{rows}\end{matrix} \rightarrow \begin{pmatrix} 5 & 15 & 17 & 7 & 0 \\ 15 & 20 & 0 & 10 & 5 \\ 4 & 9 & 0 & 7 & 19 \\ 2 & 0 & 1 & 2 & 4 \\ 5 & 5 & 10 & 25 & 0 \end{pmatrix}$$

$$\begin{matrix}\text{reducing}\\ \text{columns}\end{matrix} \begin{pmatrix} 3 & 15 & 17 & 5 & 0 \\ 13 & 20 & 0 & 8 & 5 \\ 2 & 9 & 0 & 5 & 19 \\ 0 & 0 & 1 & 0 & 4 \\ 3 & 5 & 10 & 23 & 0 \end{pmatrix}$$

$$\text{Minimum uncovered element is 2} \begin{pmatrix} 1 & 13 & 17 & 3 & 0 \\ 11 & 18 & 0 & 6 & 5 \\ 0 & 7 & 0 & 3 & 19 \\ 0 & 0 & 3 & 0 & 6 \\ 1 & 3 & 10 & 21 & 0 \end{pmatrix}$$

$$\text{Minimum uncovered element is 3} \begin{pmatrix} 1 & 10 & 17 & 0 & 0 \\ 11 & 15 & 0 & 3 & 5 \\ 0 & 4 & 0 & 0 & 19 \\ 3 & 0 & 6 & 0 & 9 \\ 1 & 0 & 10 & 18 & 0 \end{pmatrix}$$

There are two solutions

D – Z (80)	D – Y (87)	
E – X (95)	E – X (95)	
F – V (90)	F – V (90)	
G – Y (85)	G – W (83)	cost 450
H – W (100)	H – Z (95)	

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Exercise A, Question 5

Question:

	100 m	Hurdles	200 m	400 m
Ahmed	14	21	37	64
Ben	13	22	40	68
Chang	12	20	38	70
Davina	13	21	39	74

A junior school has to enter four pupils in an athletics competition comprising four events; 100 m sprint, hurdles, 200 m, 400 m. The rules are that each pupil may only enter one event and the winning team is the one whose total time for the four events is the least. The school holds trials and the table shows the time, in seconds, that each of the team members takes. Reducing rows first, use the Hungarian algorithm to determine who should participate in which event in order to minimise the total time.

Solution:

$$\begin{pmatrix} 14 & 21 & 37 & 64 \\ 13 & 22 & 40 & 68 \\ 12 & 20 & 38 & 70 \\ 13 & 21 & 39 & 74 \end{pmatrix} \rightarrow \begin{matrix}\text{reducing} \\ \text{rows}\end{matrix} \rightarrow \begin{pmatrix} 0 & 7 & 23 & 50 \\ 0 & 9 & 27 & 55 \\ 0 & 8 & 26 & 58 \\ 0 & 8 & 26 & 61 \end{pmatrix}$$

$$\begin{matrix}\text{reducing} \\ \text{columns}\end{matrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 5 \\ 0 & 1 & 3 & 8 \\ 0 & 1 & 3 & 11 \end{pmatrix}$$

$$\text{Minimum uncovered element is 1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 2 & 10 \end{pmatrix}$$

$$\text{Minimum uncovered element is 2} \begin{pmatrix} 3 & 2 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

Two solutions:

A – 400 m (64)	A – 400 m (64)	
B – 100 m (13)	B – 100 m (13)	time: 136
C – hurdles (20)	C – 200 m (38)	
D – 200 m (39)	D – hurdles (21)	

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Exercise A, Question 6

Question:

	Beech	Elm	Eucalyptus	Oak	Olive
A	153	87	62	144	76
B	162	105	87	152	88
C	159	84	75	165	79
D	145	98	63	170	85
E	149	94	70	138	82

The table shows the cost, in pounds, of purchasing trees from five local nurseries. A landscape gardener wishes to support each of these local nurseries for the year and so decides to use each nursery to supply one type of tree. He will use equal numbers of each type of tree throughout the year.

Reducing rows first, use the Hungarian algorithm to determine which type of tree should be supplied by which nursery in order to minimise the total cost.

Solution:

$$\begin{pmatrix} 153 & 87 & 62 & 144 & 76 \\ 162 & 105 & 87 & 152 & 88 \\ 159 & 84 & 75 & 165 & 79 \\ 145 & 98 & 63 & 170 & 85 \\ 149 & 94 & 70 & 138 & 82 \end{pmatrix}$$

reducing rows

$$\begin{pmatrix} 91 & 25 & 0 & 82 & 14 \\ 75 & 18 & 0 & 65 & 1 \\ 84 & 9 & 0 & 90 & 4 \\ 82 & 35 & 0 & 107 & 22 \\ 79 & 24 & 0 & 68 & 12 \end{pmatrix}$$

reducing columns

$$\begin{pmatrix} 16 & 16 & 0 & 17 & 13 \\ 0 & 9 & 0 & 0 & 0 \\ 9 & 0 & 0 & 25 & 3 \\ 7 & 26 & 0 & 42 & 21 \\ 4 & 15 & 0 & 3 & 11 \end{pmatrix}$$

Minimum uncovered element is 3

$$\begin{pmatrix} 13 & 13 & 0 & 14 & 10 \\ 0 & 9 & 3 & 0 & 0 \\ 9 & 0 & 3 & 25 & 3 \\ 4 & 23 & 0 & 39 & 18 \\ 1 & 12 & 0 & 0 & 8 \end{pmatrix}$$

Minimum uncovered element is 4

$$\begin{pmatrix} 9 & 9 & 0^* & 10 & 6 \\ 0 & 9 & 7 & 0 & 0^* \\ 9 & 0^* & 7 & 25 & 3 \\ 0^* & 19 & 0 & 35 & 14 \\ 1 & 12 & 4 & 0^* & 8 \end{pmatrix}$$

reducing rows

$$\begin{pmatrix} 91 & 25 & 0 & 82 & 14 \\ 75 & 18 & 0 & 65 & 1 \\ 84 & 9 & 0 & 90 & 4 \\ 82 & 35 & 0 & 107 & 22 \\ 79 & 24 & 0 & 68 & 12 \end{pmatrix}$$

$$\begin{array}{c} \text{reducing} \\ \text{columns} \end{array} \begin{pmatrix} 16 & 16 & 0 & 17 & 13 \\ 0 & 9 & 0 & 0 & 0 \\ 9 & 0 & 0 & 25 & 3 \\ 7 & 26 & 0 & 42 & 21 \\ 4 & 15 & 0 & 3 & 11 \end{pmatrix}$$

$$\begin{array}{c} \text{Minimum} \\ \text{uncovered} \\ \text{element is 3} \end{array} \begin{pmatrix} 13 & 13 & 0 & 14 & 10 \\ 0 & 9 & 3 & 0 & 0 \\ 9 & 0 & 3 & 25 & 3 \\ 4 & 23 & 0 & 39 & 18 \\ 1 & 12 & 0 & 0 & 8 \end{pmatrix}$$

$$\begin{array}{c} \text{Minimum} \\ \text{uncovered} \\ \text{element is 4} \end{array} \begin{pmatrix} 9 & 9 & 0^* & 10 & 6 \\ 0 & 9 & 7 & 0 & 0^* \\ 9 & 0^* & 7 & 25 & 3 \\ 0^* & 19 & 0 & 35 & 14 \\ 1 & 12 & 4 & 0^* & 8 \end{pmatrix}$$

A – Eucalyptus (62)
B – Olive (88)
C – Elm (84) Cost 517
D – Beech (145)
E – Oak (138)

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Exercise B, Question 1

Question:

The table shows the cost, in pounds, of allocating workers to tasks.
Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, show the table at each stage, and state your final solution and its cost.

	Task M	Task N
Worker J	23	26
Worker K	26	30
Worker L	29	28

Solution:

$$\begin{pmatrix} 23 & 26 & 0 \\ 26 & 30 & 0 \\ 29 & 28 & 0 \end{pmatrix} \xrightarrow[\text{columns}]{\text{reducing}} \begin{pmatrix} 0 & 0 & 0 \\ 3 & 4 & 0 \\ 6 & 2 & 0 \end{pmatrix}$$

Minimum uncovered element is 2 $\begin{pmatrix} 0 & 0 & 2 \\ 1 & 2 & 0 \\ 4 & 0 & 0 \end{pmatrix}$

Solution: J – M (23)
K – dummy
L – N (28)

cost: 51

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Exercise B, Question 2

Question:

The table shows the cost, in pounds, of allocating workers to tasks.
Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, show the table at each stage, and state your final solution and its cost.

	Task W	Task X	Task Y	Task Z
Worker A	31	43	19	35
Worker B	28	46	10	34
Worker C	24	42	13	33

Solution:

$$\begin{pmatrix} 31 & 43 & 19 & 35 \\ 28 & 46 & 10 & 34 \\ 24 & 42 & 13 & 33 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[\text{rows}]{\text{reducing}} \begin{pmatrix} 12 & 24 & 0 & 16 \\ 18 & 36 & 0 & 24 \\ 11 & 29 & 0 & 20 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Minimum uncovered element is 11 $\begin{pmatrix} 1 & 13 & 0 & 5 \\ 7 & 25 & 0 & 13 \\ 0 & 18 & 0 & 9 \\ 0 & 0 & 11 & 0 \end{pmatrix}$

Minimum uncovered element is 5 $\begin{pmatrix} 1 & 8 & 0 & 0 \\ 7 & 20 & 0 & 8 \\ 0 & 13 & 0 & 4 \\ 5 & 0 & 16 & 0 \end{pmatrix}$

Solution:
A – Z (35)
B – Y (10)
C – W (24)
dummy – X (0)

cost: 69

Alternative solution

Minimum uncovered element is 1 $\begin{pmatrix} 0 & 12 & 0 & 4 \\ 6 & 24 & 0 & 12 \\ 0 & 18 & 1 & 9 \\ 0 & 0 & 12 & 0 \end{pmatrix}$

then

Minimum uncovered element is 4 $\begin{pmatrix} 0 & 8 & 0 & 0 \\ 6 & 20 & 0 & 8 \\ 0 & 14 & 1 & 5 \\ 4 & 0 & 16 & 0 \end{pmatrix}$

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Exercise B, Question 3

Question:

The table shows the cost, in pounds, of allocating workers to tasks.
Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, show the table at each stage, and state your final solution and its cost.

	Task R	Task S	Task T
Worker W	81	45	55
Worker X	67	32	48
Worker Y	87	38	58
Worker Z	73	37	60

Solution:

$$\begin{pmatrix} 81 & 45 & 55 & 0 \\ 67 & 32 & 48 & 0 \\ 87 & 38 & 58 & 0 \\ 73 & 37 & 60 & 0 \end{pmatrix} \begin{matrix} \text{reducing} \\ \text{columns} \end{matrix} \begin{pmatrix} 14 & 13 & 7 & 0 \\ 0 & 0 & 0 & 0 \\ 20 & 6 & 10 & 0 \\ 6 & 5 & 12 & 0 \end{pmatrix}$$

Minimum uncovered element is 5 $\begin{pmatrix} 9 & 8 & 2 & 0 \\ 0 & 0 & 0 & 5 \\ 15 & 1 & 5 & 0 \\ 1 & 0 & 7 & 0 \end{pmatrix}$

Minimum uncovered element is 1 $\begin{pmatrix} 8 & 8 & 1 & 0 \\ 0 & 1 & 0 & 6 \\ 14 & 1 & 4 & 9 \\ 0 & 0 & 6 & 0 \end{pmatrix}$

Minimum uncovered element is 1 $\begin{pmatrix} 7 & 7 & 0 & 0 \\ 0 & 2 & 0 & 7 \\ 13 & 0 & 3 & 0 \\ 0 & 0 & 6 & 1 \end{pmatrix}$

There are two solutions

W – dummy   W – T (55)
X – T (48)   X – R (67)
Y – S (38)   Y – dummy   cost 159
Z – R (73)   Z – S (37)

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Exercise B, Question 4

Question:

The table shows the cost, in pounds, of allocating workers to tasks.
Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, show the table at each stage, and state your final solution and its cost.

	Task E	Task F	Task G	Task H
Worker P	24	42	32	31
Worker Q	22	39	30	35
Worker R	13	34	22	25
Worker S	19	41	27	29
Worker T	18	40	31	33

Solution:

$$\begin{pmatrix} 24 & 42 & 32 & 31 & 0 \\ 22 & 39 & 30 & 35 & 0 \\ 13 & 34 & 22 & 25 & 0 \\ 19 & 41 & 27 & 29 & 0 \\ 18 & 40 & 31 & 33 & 0 \end{pmatrix} \begin{matrix} \text{reducing} \\ \text{columns} \end{matrix} \begin{pmatrix} 11 & 8 & 10 & 6 & 0 \\ 9 & 5 & 8 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 7 & 5 & 4 & 0 \\ 5 & 6 & 9 & 8 & 0 \end{pmatrix}$$

Minimum uncovered element is 4 $\begin{pmatrix} 7 & 4 & 6 & 2 & 0 \\ 5 & 1 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 2 & 3 & 1 & 0 & 0 \\ 1 & 2 & 5 & 4 & 0 \end{pmatrix}$

<u>Either</u>

Minimum uncovered element is 1 $\begin{pmatrix} 6 & 3 & 5 & 1 & 0 \\ 4 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 2 & 3 & 1 & 0 & 1 \\ 0 & 1 & 4 & 3 & 0 \end{pmatrix}$ or $\begin{pmatrix} 6 & 3 & 5 & 2 & 0 \\ 4 & 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 1 & 5 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 4 & 0 \end{pmatrix}$

Solution:

P – dummy
Q – F (39)
R – G (22) cost 108
S – H (29)
T – E (18)

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Exercise C, Question 1

Question:

The table shows the cost, in pounds, of allocating workers to tasks.
The crosses '$\times$' indicate that that worker cannot be assigned to that task.
Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, show the table at each stage and state your final solution and its cost.

	Task L	Task M	Task N
Worker P	48	34	$\times$
Worker Q	$\times$	37	67
Worker R	53	43	56

Solution:

$$\begin{pmatrix} 48 & 34 & 140 \\ 140 & 37 & 67 \\ 53 & 43 & 56 \end{pmatrix} \begin{matrix} \text{reducing} \\ \text{rows} \end{matrix} \begin{pmatrix} 14 & 0 & 106 \\ 103 & 0 & 30 \\ 10 & 0 & 13 \end{pmatrix}$$

$$\text{reducing columns} \begin{pmatrix} 4 & 0 & 93 \\ 93 & 0 & 17 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{Minimum uncovered element is } 4 \begin{pmatrix} 0 & 0 & 89 \\ 89 & 0 & 13 \\ 0 & 4 & 0 \end{pmatrix}$$

Solution: P – L (48)
Q – M (37) cost 141
R – N (56)

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Exercise C, Question 2

Question:

The table shows the cost, in pounds, of allocating workers to tasks.
The crosses '×' indicate that that worker cannot be assigned to that task.
Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, show the table at each stage and state your final solution and its cost.

	Task D	Task E	Task F	Task G
Worker R	38	47	55	53
Worker S	32	×	47	64
Worker T	×	53	43	×
Worker U	41	48	52	47

Solution:

$$\begin{pmatrix} 38 & 47 & 55 & 53 \\ 32 & 130 & 47 & 64 \\ 130 & 53 & 43 & 130 \\ 41 & 48 & 52 & 47 \end{pmatrix} \begin{matrix}\text{reducing} \\ \text{rows}\end{matrix} \begin{pmatrix} 0 & 9 & 17 & 15 \\ 0 & 98 & 15 & 32 \\ 87 & 10 & 0 & 87 \\ 0 & 7 & 11 & 6 \end{pmatrix}$$

$$\text{reducing columns} \begin{pmatrix} 0 & 2 & 17 & 9 \\ 0 & 91 & 15 & 26 \\ 87 & 3 & 0 & 81 \\ 0 & 0 & 11 & 0 \end{pmatrix}$$

$$\text{Minimum uncovered element is 2} \begin{pmatrix} 0 & 0 & 17 & 7 \\ 0 & 89 & 15 & 24 \\ 87 & 1 & 0 & 79 \\ 2 & 0 & 13 & 0 \end{pmatrix}$$

Solution:

R – E (47)
S – D (32)
T – F (43) cost 169
U – G (47)

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Exercise C, Question 3

Question:

The table shows the cost, in pounds, of allocating workers to tasks.
The crosses ‘ × ’ indicate that that worker cannot be assigned to that task.
Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, show the table at each stage and state your final solution and its cost.

	Task P	Task Q	Task R	Task S
Worker A	46	53	67	75
Worker B	48	×	61	78
Worker C	42	46	53	62
Worker D	39	50	×	73

Solution:

$$\begin{pmatrix} 46 & 53 & 67 & 75 \\ 48 & 150 & 61 & 78 \\ 42 & 46 & 53 & 62 \\ 39 & 50 & 150 & 73 \end{pmatrix} \text{reducing rows} \begin{pmatrix} 0 & 7 & 21 & 29 \\ 0 & 102 & 13 & 30 \\ 0 & 4 & 11 & 20 \\ 0 & 11 & 111 & 34 \end{pmatrix}$$

$$\text{reducing columns} \begin{pmatrix} 0 & 3 & 10 & 9 \\ 0 & 98 & 2 & 10 \\ 0 & 0 & 0 & 0 \\ 0 & 7 & 100 & 14 \end{pmatrix}$$

$$\text{Minimum uncovered element is 2} \begin{pmatrix} 0 & 1 & 8 & 7 \\ 0 & 96 & 0 & 8 \\ 2 & 0 & 0 & 0 \\ 0 & 5 & 98 & 12 \end{pmatrix}$$

$$\text{Minimum uncovered element is 1} \begin{pmatrix} 0 & 0 & 8 & 6 \\ 0 & 95 & 0 & 7 \\ 3 & 0 & 1 & 0 \\ 0 & 4 & 98 & 11 \end{pmatrix}$$

Solution: A – Q (53)
B – R (61)
C – S (62)
D – P (39)

cost 215

Solutionbank D2
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Exercise C, Question 4

Question:

The table shows the cost, in pounds, of allocating workers to tasks.
The crosses '×' indicate that that worker cannot be assigned to that task.
Reducing rows first, use the Hungarian algorithm to find an allocation that minimises the cost. You should make your method clear, show the table at each stage and state your final solution and its cost.

	Task R	Task S	Task T	Task U	Task V
Worker J	143	112	149	137	×
Worker K	149	106	153	115	267
Worker L	137	109	143	121	×
Worker M	157	×	×	134	290
Worker N	126	101	132	111	253

Solution:

$$\begin{pmatrix} 143 & 112 & 149 & 137 & 600 \\ 149 & 106 & 153 & 115 & 267 \\ 137 & 109 & 143 & 121 & 600 \\ 157 & 600 & 600 & 134 & 290 \\ 126 & 101 & 132 & 111 & 253 \end{pmatrix} \begin{array}{c} \text{reducing} \\ \text{rows} \end{array} \begin{pmatrix} 31 & 0 & 37 & 25 & 488 \\ 43 & 0 & 47 & 9 & 161 \\ 28 & 0 & 34 & 12 & 491 \\ 23 & 466 & 466 & 0 & 156 \\ 25 & 0 & 31 & 10 & 152 \end{pmatrix}$$

$$\text{reducing columns} \begin{pmatrix} 8 & 0 & 6 & 25 & 336 \\ 20 & 0 & 16 & 9 & 9 \\ 5 & 0 & 3 & 12 & 339 \\ 0 & 466 & 435 & 0 & 4 \\ 2 & 0 & 0 & 10 & 0 \end{pmatrix}$$

$$\text{Minimum uncovered element is 3} \begin{pmatrix} 5 & 0 & 3 & 22 & 333 \\ 17 & 0 & 13 & 6 & 6 \\ 2 & 0 & 0 & 9 & 336 \\ 0 & 469 & 435 & 0 & 4 \\ 2 & 3 & 0 & 10 & 0 \end{pmatrix}$$

$$\text{Minimum uncovered element is 3} \begin{pmatrix} 2 & 0 & 0 & 19 & 330 \\ 14 & 0 & 10 & 3 & 3 \\ 2 & 3 & 0 & 9 & 336 \\ 0 & 472 & 435 & 0 & 4 \\ 2 & 6 & 0 & 10 & 0 \end{pmatrix}$$

$$\text{Minimum uncovered element is 2} \begin{pmatrix} 0 & 0 & 0 & 17 & 328 \\ 12 & 0 & 10 & 1 & 1 \\ 0 & 3 & 0 & 7 & 334 \\ 0 & 474 & 437 & 0 & 4 \\ 2 & 8 & 2 & 10 & 0 \end{pmatrix}$$

Solution:

J – R (143)
K – S (106)
L – T (143)
M – U (134)
N – V (253)

cost 779

Solutionbank D2
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Exercise D, Question 1

Question:

The table shows the **profit**, in pounds, of allocating workers to tasks.
Reducing rows first, use the Hungarian algorithm to find an allocation that maximises the profit. You should make your method clear, show the table at each stage and state your final solution and its profit.

	Task C	Task D	Task E
Worker L	37	15	12
Worker M	25	13	16
Worker N	32	41	35

Solution:

$$\begin{pmatrix} 37 & 15 & 12 \\ 25 & 13 & 16 \\ 32 & 41 & 35 \end{pmatrix} \text{ Subtracting all terms from 41 } \begin{pmatrix} 4 & 26 & 29 \\ 16 & 28 & 25 \\ 9 & 0 & 6 \end{pmatrix}$$

$$\text{reducing rows } \begin{pmatrix} 0 & 22 & 25 \\ 0 & 12 & 9 \\ 9 & 0 & 6 \end{pmatrix} \text{ reducing columns } \begin{pmatrix} 0 & 22 & 19 \\ 0 & 12 & 3 \\ 9 & 0 & 0 \end{pmatrix}$$

$$\text{Minimum uncovered element is 3 } \begin{pmatrix} 0 & 19 & 16 \\ 0 & 9 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

Solution L − C(37)
M − E(16)
N − D(41)

Profit 94

Solutionbank D2

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Exercise D, Question 2

Question:

The table shows the **profit**, in pounds, of allocating workers to tasks.
Reducing rows first, use the Hungarian algorithm to find an allocation that maximises the profit. You should make your method clear, show the table at each stage and state your final solution and its profit.

	Task S	Task T	Task U	Task V
Worker C	36	34	32	35
Worker D	37	32	34	33
Worker E	42	35	37	36
Worker F	39	34	35	35

Solution:

$$\begin{pmatrix} 36 & 34 & 32 & 35 \\ 37 & 32 & 34 & 33 \\ 42 & 35 & 37 & 36 \\ 39 & 34 & 35 & 35 \end{pmatrix} \text{ Subtracting all terms from 42 } \begin{pmatrix} 6 & 8 & 10 & 7 \\ 5 & 10 & 8 & 9 \\ 0 & 7 & 5 & 6 \\ 3 & 8 & 7 & 7 \end{pmatrix}$$

$$\text{reducing rows } \begin{pmatrix} 0 & 2 & 4 & 1 \\ 0 & 5 & 3 & 4 \\ 0 & 7 & 5 & 6 \\ 0 & 5 & 4 & 4 \end{pmatrix} \text{ reducing columns } \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 3 \\ 0 & 5 & 2 & 5 \\ 0 & 3 & 1 & 3 \end{pmatrix}$$

$$\text{Minimum uncovered element is 3 } \begin{pmatrix} 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

There are two solutions

C – T (34)		C – V (35)		
D – U (34)	or	D – U (34)		Profit 145
E – S (42)		E – S (42)		
F – V (35)		F – T (34)		

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Exercise D, Question 3

Question:

The table shows the **profit**, in pounds, of allocating workers to tasks.
Reducing rows first, use the Hungarian algorithm to find an allocation that maximises the profit. You should make your method clear, show the table at each stage and state your final solution and its profit.

	Task E	Task F	Task G	Task H
Worker R	20	22	14	24
Worker S	20	19	12	20
Worker T	13	10	18	16
Worker U	22	23	9	28

Solution:

$$\begin{pmatrix} 20 & 22 & 14 & 24 \\ 20 & 19 & 12 & 20 \\ 13 & 10 & 18 & 16 \\ 22 & 23 & 9 & 28 \end{pmatrix} \text{ Subtracting all terms from 28 } \begin{pmatrix} 8 & 6 & 14 & 4 \\ 8 & 9 & 16 & 8 \\ 15 & 18 & 10 & 12 \\ 6 & 5 & 19 & 0 \end{pmatrix}$$

$$\text{reducing rows } \begin{pmatrix} 4 & 2 & 10 & 0 \\ 0 & 1 & 8 & 0 \\ 5 & 8 & 0 & 2 \\ 6 & 5 & 19 & 0 \end{pmatrix} \text{ reducing columns } \begin{pmatrix} 4 & 1 & 10 & 0 \\ 0 & 0 & 8 & 0 \\ 5 & 7 & 0 & 2 \\ 6 & 4 & 19 & 0 \end{pmatrix}$$

$$\text{Minimum uncovered element is 1 } \begin{pmatrix} 3 & 0 & 9 & 0 \\ 0 & 0 & 8 & 1 \\ 5 & 7 & 0 & 3 \\ 5 & 3 & 18 & 0 \end{pmatrix}$$

Solution R – F (22)
S – E (20)
T – G (18)
U – H (28)

Profit 88

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Exercise D, Question 4

Question:

The table shows the **profit**, in pounds, of allocating workers to tasks.
Reducing rows first, use the Hungarian algorithm to find an allocation that maximises the profit. You should make your method clear, show the table at each stage and state your final solution and its profit.

	Task J	Task K	Task L	Task M	Task N
Worker A	85	95	86	87	97
Worker B	110	111	95	115	100
Worker C	90	95	86	93	105
Worker D	85	87	84	85	87
Worker E	100	100	105	120	95

Solution:

$$\begin{pmatrix} 85 & 95 & 86 & 87 & 97 \\ 110 & 111 & 95 & 115 & 100 \\ 90 & 95 & 86 & 93 & 105 \\ 85 & 87 & 84 & 85 & 87 \\ 100 & 100 & 105 & 120 & 95 \end{pmatrix} \text{ Subtracting all terms from 120 } \begin{pmatrix} 35 & 25 & 34 & 33 & 23 \\ 10 & 9 & 25 & 5 & 20 \\ 30 & 25 & 34 & 27 & 15 \\ 35 & 33 & 36 & 35 & 33 \\ 20 & 20 & 15 & 0 & 25 \end{pmatrix}$$

$$\text{reducing rows } \begin{pmatrix} 12 & 2 & 11 & 10 & 0 \\ 5 & 4 & 20 & 0 & 15 \\ 15 & 10 & 19 & 12 & 0 \\ 2 & 0 & 3 & 2 & 0 \\ 20 & 20 & 15 & 0 & 25 \end{pmatrix} \text{ reducing columns } \begin{pmatrix} 10 & 2 & 8 & 10 & 0 \\ 3 & 4 & 17 & 0 & 15 \\ 13 & 10 & 16 & 12 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 18 & 20 & 12 & 0 & 25 \end{pmatrix}$$

$$\text{Minimum uncovered element is 2 } \begin{pmatrix} 8 & 0 & 6 & 10 & 0 \\ 1 & 2 & 15 & 0 & 15 \\ 11 & 8 & 14 & 12 & 0 \\ 0 & 0 & 0 & 4 & 2 \\ 16 & 18 & 10 & 0 & 25 \end{pmatrix}$$

$$\text{Minimum uncovered element is 1 } \begin{pmatrix} 7 & 0 & 5 & 10 & 0 \\ 0 & 2 & 14 & 0 & 15 \\ 10 & 8 & 13 & 12 & 0 \\ 0 & 1 & 0 & 5 & 3 \\ 15 & 18 & 9 & 0 & 25 \end{pmatrix}$$

Solution A – K (95)
B – J (110)
C – N (105)
D – L (84)
E – M (120)

Profit 514

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Exercise E, Question 1

Question:

The table shows the cost, in pounds, of allocating workers to tasks.
You wish to minimise the total cost.

Formulate this as a linear programming problem, defining your variables and making the objective and constraints clear.

	Task C	Task D	Task E
Worker L	37	15	12
Worker M	25	13	16
Worker N	32	41	35

Solution:

Let x_{ij} be 0 or 1

$$x_{ij}\begin{cases}1 \text{ if worker, } i \text{ does task } j\\ 0 \text{ otherwise}\end{cases}$$

where $i \in \{\mathrm{L},\mathrm{M},\mathrm{N}\}$ and $j \in \{\mathrm{C},\mathrm{D},\mathrm{E}\}$

$$\begin{aligned}\text{Minimise } C = {} & 37x_{\mathrm{LC}} + 15x_{\mathrm{LD}} + 12x_{\mathrm{LE}}\\ & + 25x_{\mathrm{MC}} + 13x_{\mathrm{MD}} + 16x_{\mathrm{ME}}\\ & + 32x_{\mathrm{NC}} + 41x_{\mathrm{ND}} + 35x_{\mathrm{NE}}\end{aligned}$$

Subject to:

$$\begin{aligned}\sum x_{\mathrm{L}j} &= 1\\ \sum x_{\mathrm{M}j} &= 1\\ \sum x_{\mathrm{N}j} &= 1\\ \sum x_{i\mathrm{C}} &= 1\\ \sum x_{i\mathrm{D}} &= 1\\ \sum x_{i\mathrm{E}} &= 1\end{aligned}$$

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Exercise E, Question 2

Question:

The table shows the cost, in pounds, of allocating workers to tasks.
You wish to minimise the total cost.

Formulate this as a linear programming problem, defining your variables and making the objective and constraints clear.

	Task S	Task T	Task U	Task V
Worker C	36	34	32	35
Worker D	37	32	34	33
Worker E	42	35	37	36
Worker F	39	34	35	35

Solution:

Let x_{ij} be 0 or 1

$$x_{ij}\begin{cases}1 \text{ if worker } i \text{ does task } j\\ 0 \text{ otherwise}\end{cases}$$

where $i \in \{C, D, E, F\}$ and $j \in \{S, T, U, V\}$.

$$\begin{aligned}\text{Minimise } C = {} & 36x_{CS} + 34x_{CT} + 32x_{CU} + 35x_{CV}\\ & +37x_{DS} + 32x_{DT} + 34x_{DU} + 33x_{DV}\\ & +42x_{ES} + 35x_{ET} + 37x_{EU} + 36x_{EV}\\ & +39x_{FS} + 34x_{FT} + 35x_{FU} + 35x_{FV}\end{aligned}$$

Subject to:
$$\sum x_{Cj} = 1$$
$$\sum x_{Dj} = 1$$
$$\sum x_{Ej} = 1$$
$$\sum x_{Fj} = 1$$
$$\sum x_{iS} = 1$$
$$\sum x_{iT} = 1$$
$$\sum x_{iU} = 1$$
$$\sum x_{iV} = 1$$

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Exercise E, Question 3

Question:

Repeat question 1, but take the entries to be the **profit** earned in allocating workers to tasks, and seek to maximise the total profit.

Solution:

Subtract all terms from 41 $\begin{pmatrix} 4 & 26 & 29 \\ 16 & 28 & 25 \\ 9 & 0 & 6 \end{pmatrix}$

Let x_{ij} be 0 or 1

$x_{ij} \begin{cases} 1 & \text{if worker; } i \text{ does task } j \\ 0 & \text{otherwise} \end{cases}$

where $i \in \{L, M, N\}$ and $j \in \{C, D, E\}$

Minimise $P = 4x_{LC} + 26x_{LD} + 29x_{LE}$
$+16x_{MC} + 28x_{MD} + 25x_{ME}$
$+9x_{NC} + 6x_{NE}$

Subject to:

$\sum x_{Lj} = 1 \qquad \sum x_{iC} = 1$

$\sum x_{Mj} = 1 \qquad \sum x_{iD} = 1$

$\sum x_{Nj} = 1 \qquad \sum x_{iE} = 1$

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Exercise E, Question 4

Question:

Repeat question 2, but take the entries to be the **profit** earned in allocating workers to tasks, and seek to maximise the total profit.

Solution:

Subtract all terms from 42

$$\begin{pmatrix} 6 & 8 & 10 & 7 \\ 5 & 10 & 8 & 9 \\ 0 & 7 & 5 & 6 \\ 3 & 8 & 7 & 7 \end{pmatrix}$$

Let x_{ij} be 0 or 1

$$x_{ij} \begin{cases} 1 & \text{if worker } i \text{ does task } j \\ 0 & \text{otherwise} \end{cases}$$

where $i \in \{C, D, E, F\}$ and $j \in \{S, T, U, V\}$

$$\text{Minimise } P = 6x_{CS} + 8x_{CT} + 10x_{CU} + 7x_{CV}$$
$$5x_{DS} + 10x_{DT} + 8x_{DU} + 9x_{DV}$$
$$+7x_{ET} + 5x_{EU} + 6x_{EV}$$
$$3x_{FS} + 8x_{FT} + 7x_{FU} + 7x_{FV}$$

Subject to:

$\sum x_{Cj} = 1$ $\quad \sum x_{iS} = 1$

$\sum x_{Dj} = 1$ $\quad \sum x_{iT} = 1$

$\sum x_{Ej} = 1$ $\quad \sum x_{iU} = 1$

$\sum x_{Fj} = 1$ $\quad \sum x_{iV} = 1$

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Exercise F, Question 1

Question:

	Airport	Depot	Docks	Station
Bring-it	322	326	326	328
Collect-it	318	325	324	325
Fetch-it	315	319	317	320
Haul-it	323	322	319	321

A museum is staging a special exhibition. They have been loaned exhibits from other museums and from private collectors. Seven days before the exhibition starts these exhibits will be arriving at the airport, road depot, docks and railway station and in each case the single load has to be transported to the museum. There are four local companies that could deliver the exhibits: Bring-it, Collect-it, Fetch-it and Haul-it. Since all four companies are helping to sponsor the exhibition, the museum wishes to use all four companies, allocating each company to just one arrival point.

The table shows the cost, in pounds, of using each company for each task. The museum wishes to minimise its transportation costs.

Reducing rows first, use the Hungarian algorithm to determine the allocation that minimises the total cost. You must make your method clear and show the table after each stage. State your final allocation and its cost.

Solution:

$$\begin{pmatrix} 322 & 326 & 326 & 328 \\ 318 & 325 & 324 & 325 \\ 315 & 319 & 317 & 320 \\ 323 & 322 & 319 & 321 \end{pmatrix} \text{ reducing rows } \begin{pmatrix} 0 & 4 & 4 & 6 \\ 0 & 7 & 6 & 7 \\ 0 & 4 & 2 & 5 \\ 4 & 3 & 0 & 2 \end{pmatrix}$$

$$\text{reducing columns } \begin{pmatrix} 0 & 1 & 4 & 4 \\ 0 & 4 & 6 & 5 \\ 0 & 1 & 2 & 3 \\ 4 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Minimum uncovered element is 1 } \begin{pmatrix} 0 & 0 & 3 & 3 \\ 0 & 3 & 5 & 4 \\ 0 & 0 & 1 & 2 \\ 5 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Minimum uncovered element is 1 } \begin{pmatrix} 0 & 0 & 2 & 2 \\ 0 & 3 & 4 & 3 \\ 0 & 0 & 0 & 1 \\ 6 & 1 & 0 & 0 \end{pmatrix}$$

Bring-it – Depot (326)
Collect-it – Airport (318)
Fetch-it – Docks (317)
Haul-it – Station (321)

Cost 1282

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Exercise F, Question 2

Question:

	Back	Breast	Butterfly	Crawl
Jack	18	20	19	14
Kyle	19	21	19	14
Liam	17	20	20	16
Mike	20	21	20	15

A medley relay swimming team consists of four swimmers. The first member of the team swims one length of backstroke, then the second person swims a length of breaststroke, then the next a length of butterfly and finally the fourth person a length of crawl. Each member of the team must swim just one length. All the team members could swim any of the lengths, but some members of the team are faster at one or two particular strokes.

The table shows the time, in seconds, each member of the team took to swim each length using each type of stroke during the last training session

a Use the Hungarian algorithm, reducing rows first, to find an allocation that minimises the total time it takes the team to complete all four lengths.

b State the best time in which this team could complete the race.

c Show that there is more than one way of allocating the team so that they can achieve this best time.

In fact there are four optimal solutions to this problem.

Solution:

a

$$\begin{pmatrix} 18 & 20 & 19 & 14 \\ 19 & 21 & 19 & 14 \\ 17 & 20 & 20 & 16 \\ 20 & 21 & 20 & 15 \end{pmatrix} \xrightarrow[\text{rows}]{\text{reducing}} \begin{pmatrix} 4 & 6 & 5 & 0 \\ 5 & 7 & 5 & 0 \\ 1 & 4 & 4 & 0 \\ 5 & 6 & 5 & 0 \end{pmatrix}$$

$$\begin{matrix}\text{reducing} \\ \text{columns}\end{matrix} \begin{pmatrix} 3 & 2 & 1 & 0 \\ 4 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 \end{pmatrix}$$

Minimum uncovered element is 1 $\begin{pmatrix} 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{pmatrix}$

Minimum uncovered element is 1 $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 \end{pmatrix}$

b and **c**

There are 4 solutions each of duration 71 seconds

J – Br (20)	J – C (14)	J – Br (20)	J – Bu (19)
K – Bu (19)	K – Bu (19)	K – Cr (14)	K – Cr (14)
L – Ba (17)	L – Ba (17)	L – Ba (17)	L – Ba (17)
M – C (15)	M – Br (21)	M – Bu (20)	M – Br (21)

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Exercise F, Question 3

Question:

	Grand Hall	Dining Room	Gallery	Bedroom	Kitchen
Alf	8	19	11	14	12
Betty	12	17	14	18	20
Charlie	10	22	18	14	19
Donna	9	15	16	15	21
Eve	14	23	20	20	19

Five tour guides work at Primkal Mansion. They talk to groups of tourists about five particularly significant rooms. Each tour guide will be stationed in a particular room for the day, but may change rooms the next day. The tourists will listen to each talk before moving on to the next room. Once they have listened to all five talks they will head off to the gift shop.

The table shows the average length of each tour guide's talk in each room.

A tourist party arrives at the Mansion.

a Use the Hungarian algorithm, reducing rows first, to find the **quickest** time that the tour could take. You should state the optimal allocation and its length and show the state of the table at each stage.

b Adapt the table and re-apply the Hungarian algorithm, reducing rows first, to find the **longest** time that the tour could take. You should state the optimal allocation and its duration and show the state of the table at each stage.

Solution:

a

$$\begin{pmatrix} 8 & 19 & 11 & 14 & 12 \\ 12 & 17 & 14 & 18 & 20 \\ 10 & 22 & 18 & 14 & 19 \\ 9 & 15 & 16 & 15 & 21 \\ 14 & 23 & 20 & 20 & 19 \end{pmatrix} \begin{matrix} \text{reducing} \\ \text{rows} \end{matrix} \begin{pmatrix} 0 & 11 & 3 & 6 & 4 \\ 0 & 5 & 2 & 5 & 8 \\ 0 & 12 & 8 & 7 & 9 \\ 0 & 6 & 7 & 6 & 12 \\ 0 & 9 & 6 & 6 & 5 \end{pmatrix}$$

$$\begin{matrix} \text{reducing} \\ \text{columns} \end{matrix} \begin{pmatrix} 0 & 6 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 7 & 6 & 0 & 5 \\ 0 & 1 & 5 & 2 & 8 \\ 0 & 4 & 4 & 2 & 1 \end{pmatrix}$$

$$\text{Minimum uncovered element is 1} \begin{pmatrix} 0 & 5 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 & 5 \\ 0 & 6 & 5 & 0 & 5 \\ 0 & 0 & 4 & 2 & 8 \\ 0 & 3 & 3 & 2 & 1 \end{pmatrix}$$

Solution
Aif - Kitchen (12)
Betty - Gallery (14)
Charlie - Bedroom (14)
Donna - Dining room (15)
Eve - Grand Hall (14) Minimum time 69 minutes

b

$$\text{Subtracting all terms from 23} \begin{pmatrix} 15 & 4 & 12 & 9 & 11 \\ 11 & 6 & 9 & 5 & 3 \\ 13 & 1 & 5 & 9 & 4 \\ 14 & 8 & 7 & 8 & 2 \\ 9 & 0 & 3 & 3 & 4 \end{pmatrix}$$

$$\begin{matrix} \text{reducing} \\ \text{rows} \end{matrix} \begin{pmatrix} 11 & 0 & 8 & 5 & 7 \\ 8 & 3 & 6 & 2 & 0 \\ 12 & 0 & 4 & 8 & 3 \\ 12 & 6 & 5 & 6 & 0 \\ 9 & 0 & 3 & 3 & 4 \end{pmatrix} \begin{matrix} \text{reducing} \\ \text{columns} \end{matrix} \begin{pmatrix} 3 & 0 & 5 & 3 & 7 \\ 0 & 3 & 3 & 0 & 0 \\ 4 & 0 & 1 & 6 & 3 \\ 4 & 6 & 2 & 4 & 0 \\ 1 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\text{Minimum uncovered element is 1} \begin{pmatrix} 2 & 0 & 5 & 2 & 6 \\ 0 & 4 & 4 & 0 & 0 \\ 3 & 0 & 1 & 5 & 2 \\ 4 & 7 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Minimum uncovered element is 1 $\begin{pmatrix} 1 & 0 & 4 & 1 & 6 \\ 0 & 5 & 4 & 0 & 1 \\ 2 & 0 & 0 & 4 & 2 \\ 3 & 7 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 4 \end{pmatrix}$

Solutions

Alf – Dining room (19)		Alf – Dining room (19)
Betty – Hall (12)		Betty – Bedroom (15)
Charlie – Gallery (18)	or	Charlie – Gallery (18)
Donna – Kitchen (21)		Donna – Kitchen (21)
Eve – Bedroom (20)		Eve – Hall (14)

Maximum time 90 minutes

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Exercise F, Question 4

Question:

	Award ceremony	Film premiere	Celebrity party
Denzel	245	378	459
Eun-Ling	250	387	467
Frank	224	350	442
Gabby	231	364	453

A company hires out chauffer-driven, luxury stretch-limousines. They have to provide cars for three events next Saturday night: an award ceremony, a film premiere and a celebrity party. The company has four chauffeurs available and the cost, in pounds, of assigning each of them to each event is shown in the table above. The company wishes to minimise its total costs.

a Explain why it is necessary to add a dummy event.

b Reducing rows first, use the Hungarian algorithm to determine the allocation that minimises the total cost. You should state the optimal allocation and its cost and show the state of the table at each stage.

Solution:

a There are 4 chauffeurs but only 3 tasks.

b
$$\begin{pmatrix} 245 & 378 & 459 & 0 \\ 250 & 387 & 467 & 0 \\ 224 & 350 & 442 & 0 \\ 231 & 364 & 453 & 0 \end{pmatrix} \begin{matrix} \text{reducing} \\ \text{columns} \end{matrix} \begin{pmatrix} 21 & 28 & 17 & 0 \\ 26 & 37 & 25 & 0 \\ 0 & 0 & 0 & 0 \\ 7 & 14 & 11 & 0 \end{pmatrix}$$

Minimum uncovered element is 7
$$\begin{pmatrix} 14 & 21 & 10 & 0 \\ 19 & 30 & 18 & 0 \\ 0 & 0 & 0 & 7 \\ 0 & 7 & 4 & 0 \end{pmatrix}$$

Minimum uncovered element is 10
$$\begin{pmatrix} 4 & 11 & 0 & 0 \\ 9 & 20 & 8 & 0 \\ 0 & 0 & 0 & 17 \\ 0 & 7 & 4 & 10 \end{pmatrix}$$

Solution:

D – Party (459)

E – Dummy

F – Film (350)

G – Award (231)

Cost £1040

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Exercise F, Question 5

Question:

	Catering	Cleaning	Computer	Copying	Post
Blue	No	863	636	628	739
Green	562	796	583	478	674
Orange	No	825	672	583	756
Red	635	881	650	538	No
Yellow	688	934	No	554	No

A large office block is to be serviced and supplied by five companies Blue supplies, Green services, Orange office supplies, Red Co and Yellow Ltd. These companies have each applied to take care of catering, cleaning, computer supplies/servicing, copying and postal services.

The table shows the daily cost of using each firm, in pounds.

For political reasons the owners of the office block will use all five companies, one for each of the five tasks.

Some of the companies cannot offer some services and this is indicated by 'No'.

Use the Hungarian algorithm, reducing rows first, to allocate the companies to the services in such a way as to minimise the total cost. You should state the optimal allocation and its cost and show the state of the table at each stage.

Solution:

$$\begin{pmatrix} 2000 & 863 & 636 & 628 & 739 \\ 562 & 796 & 583 & 478 & 674 \\ 2000 & 825 & 672 & 583 & 756 \\ 635 & 881 & 650 & 538 & 2000 \\ 688 & 934 & 2000 & 554 & 2000 \end{pmatrix} \begin{matrix} \text{reducing} \\ \text{rows} \end{matrix} \begin{pmatrix} 1372 & 235 & 8 & 0 & 111 \\ 84 & 318 & 105 & 0 & 196 \\ 1417 & 242 & 89 & 0 & 173 \\ 97 & 343 & 112 & 0 & 1462 \\ 134 & 380 & 1446 & 0 & 1446 \end{pmatrix}$$

$$\begin{matrix} \text{reducing} \\ \text{columns} \end{matrix} \begin{pmatrix} 1288 & 0 & 0 & 0 & 0 \\ 0 & 83 & 97 & 0 & 85 \\ 1333 & 7 & 81 & 0 & 62 \\ 13 & 108 & 104 & 0 & 1351 \\ 50 & 145 & 1433 & 0 & 1335 \end{pmatrix}$$

$$\text{Minimum uncovered element is 7} \begin{pmatrix} 1288 & 0 & 0 & 7 & 0 \\ 0 & 83 & 97 & 7 & 85 \\ 1326 & 0 & 74 & 0 & 55 \\ 6 & 101 & 97 & 0 & 1344 \\ 43 & 138 & 1426 & 0 & 1328 \end{pmatrix}$$

$$\text{Minimum uncovered element is 83} \begin{pmatrix} 1371 & 0 & 0 & 90 & 0 \\ 0 & 0 & 14 & 13 & 2 \\ 1409 & 0 & 74 & 83 & 55 \\ 6 & 18 & 14 & 0 & 1261 \\ 43 & 55 & 1343 & 0 & 1245 \end{pmatrix}$$

$$\text{Minimum uncovered element is 6} \begin{pmatrix} 1371 & 6 & 0 & 96 & 0 \\ 0 & 6 & 14 & 19 & 2 \\ 1403 & 0 & 68 & 83 & 49 \\ 0 & 18 & 8 & 0 & 1255 \\ 37 & 55 & 1337 & 0 & 1239 \end{pmatrix}$$

$$\text{Minimum uncovered element is 2} \begin{pmatrix} 1373 & 6 & 0 & 98 & 0 \\ 0 & 4 & 12 & 19 & 0 \\ 1405 & 0 & 68 & 85 & 49 \\ 0 & 16 & 6 & 0 & 1253 \\ 37 & 53 & 1335 & 0 & 1237 \end{pmatrix}$$

Blue – Computer (636)
Green – Post (674)
Orange – Cleaning (825)
Red – Catering (635)
Yellow – Copying (554)
Cost £3324

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Exercise F, Question 6

Question:

	Cafe	Coffee shop	Restaurant	Snack shop
Ghost train	834	365	580	648
Log flume	874	375	No	593
Roller coaster	743	289	No	665
Teddie's adventure	899	500	794	No

The owners of a theme park wish to provide a café, coffee shop, restaurant and snack shop at four sites: next to the ghost train, log flume, roller coaster and teddie's adventure. They employ a market researcher who estimates the daily profit of each type of catering at each site.

The market researcher also suggests that some types of catering are not suitable at some of the sites, these are indicated by 'No'.

Using the Hungarian algorithm, determine the allocation that provides the maximum daily profit.

This question is a maximising question and one with incomplete data. You need to chose numbers to put at the sites marked 'No' so that they become 'unattractive' to the algorithm after it has been altered to look for the maximum solution.

Solution:

$$\begin{pmatrix} 834 & 365 & 580 & 648 \\ 874 & 375 & 2000 & 593 \\ 743 & 289 & 2000 & 665 \\ 899 & 500 & 794 & 2000 \end{pmatrix} \begin{matrix} \text{Subtracting} \\ \text{all terms} \\ \text{from 1000} \end{matrix} \begin{pmatrix} 166 & 635 & 420 & 352 \\ 126 & 625 & 1000 & 407 \\ 257 & 711 & 1000 & 335 \\ 101 & 500 & 206 & 1000 \end{pmatrix}$$

$$\begin{matrix} \text{reducing} \\ \text{rows} \end{matrix} \begin{pmatrix} 0 & 469 & 254 & 186 \\ 0 & 499 & 874 & 281 \\ 0 & 454 & 743 & 78 \\ 0 & 399 & 105 & 899 \end{pmatrix} \begin{matrix} \text{reducing} \\ \text{columns} \end{matrix} \begin{pmatrix} 0 & 70 & 149 & 108 \\ 0 & 100 & 769 & 203 \\ 0 & 55 & 638 & 0 \\ 0 & 0 & 0 & 821 \end{pmatrix}$$

$$\text{Minimum uncovered element is 55} \begin{pmatrix} 0 & 15 & 94 & 108 \\ 0 & 45 & 714 & 203 \\ 0 & 0 & 583 & 0 \\ 55 & 0 & 0 & 876 \end{pmatrix}$$

$$\text{Minimum uncovered element is 15} \begin{pmatrix} 0 & 0 & 79 & 93 \\ 0 & 30 & 699 & 188 \\ 15 & 0 & 583 & 0 \\ 70 & 0 & 0 & 876 \end{pmatrix}$$

Ghost train – Coffee shop (365)
Log flume – Cafe (874)
Roller coaster – Snack shop (665)
Teddie's adventure – Restaurant (794)
Profit 2698

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Exercise F, Question 7

Question:

	1	2	3	4
P	143	243	247	475
Q	132	238	218	437
R	126	207	197	408
S	138	222	238	445

Four workers P, Q, R and S are to be assigned to four tasks 1, 2, 3 and 4. Each worker is to be assigned to one task and each task must be assigned to one worker. The cost, in pounds, of using each worker for each task is given in the table above. The cost is to be minimised.

Formulate this as a linear programming problem, defining your variables and making the objective and constraints clear.

Solution:

Let x_{ij} be 0 or 1

$$x_{ij}\begin{cases}1 \text{ if worker } i \text{ does task } j\\ 0 \text{ otherwise}\end{cases}$$

where $i \in \{P, Q, R, S\}$ and $j \in \{1, 2, 3, 4\}$

$$\begin{aligned}\text{Minimise } C = {} & 143x_{P1} + 243x_{P2} + 247x_{P3} + 475x_{P4}\\ & +132x_{Q1} + 238x_{Q2} + 218x_{Q3} + 437x_{Q4}\\ & +126x_{R1} + 207x_{R2} + 197x_{R3} + 408x_{R4}\\ & +138x_{S1} + 222x_{S2} + 238x_{S3} + 445x_{S4}\end{aligned}$$

Subject to: $\sum x_{Pj} = 1$

$$\sum x_{Pj} = 1$$
$$\sum x_{Qj} = 1$$
$$\sum x_{Rj} = 1$$
$$\sum x_{Sj} = 1$$
$$\sum x_{i1} = 1$$
$$\sum x_{i2} = 1$$
$$\sum x_{i3} = 1$$
$$\sum x_{i4} = 1$$

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Exercise F, Question 8

Question:

	A	B	C	D
P	13	17	15	18
Q	15	19	12	19
R	16	20	13	22
S	14	15	17	24

Krunchy Cereals Ltd will send four salesman P, Q, R and S to visit four store managers at A, B, C and D to take orders for their new products. Each salesman will visit only one store manager and each store manager will be visited by just one salesman. The expected value, in thousands of pounds, of the orders won is shown in the table above. The company wishes to maximise the value of the orders.

Formulate this as a linear programming problem, defining your variables and making the objective and constraints clear.

Solution:

$$\begin{pmatrix} 13 & 17 & 15 & 18 \\ 15 & 19 & 12 & 19 \\ 16 & 20 & 13 & 22 \\ 14 & 15 & 17 & 24 \end{pmatrix} \text{ Subtracting all terms from 24 } \begin{pmatrix} 11 & 7 & 9 & 6 \\ 9 & 5 & 12 & 5 \\ 8 & 4 & 11 & 2 \\ 10 & 9 & 7 & 0 \end{pmatrix}$$

Let x_{ij} be 0 or 1

$$x_{ij} \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \text{ otherwise} \end{cases}$$

where $i \in \{P, Q, R, S\}$ and $j \in \{A, B, C, D\}$

$$\begin{aligned} \text{Minimise } P = {} & 11x_{PA} + 7x_{PB} + 9x_{PC} + 6x_{PD} \\ & +9x_{QA} + 5x_{QB} + 12x_{QC} + 5x_{QD} \\ & +8x_{RA} + 4x_{RB} + 11x_{RC} + 2x_{RD} \\ & +10x_{SA} + 9x_{SB} + 7x_{SC} \end{aligned}$$

$$\begin{aligned} \text{Subject to: } & \sum x_{Pj} = 1 \\ & \sum x_{Qj} = 1 \\ & \sum x_{Rj} = 1 \\ & \sum x_{Sj} = 1 \\ & \sum x_{iA} = 1 \\ & \sum x_{iB} = 1 \\ & \sum x_{iC} = 1 \\ & \sum x_{iD} = 1 \end{aligned}$$

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Exercise A, Question 1

Question:

Complete the table of least distances for the network. State the route you used for each of your entries.

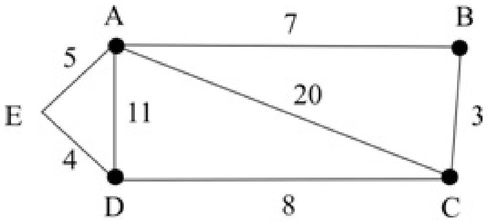

Table of least differences

	A	B	C	D	E
A	–	7			5
B	7	–	3		
C		3	–	8	12
D			8	–	4
E	5		12	4	–

Solution:

	A	B	C	D	E
A	–	7	**10**	**9**	5
B	7	–	3	**11**	**12**
C	**10**	3	–	8	12
D	**9**	**11**	8	–	4
E	5	**12**	12	4	–

AC – the shortest route is ABC length 10
AD – the shortest route is AED length 9
BD – the shortest route is BCD length 11
BE – the shortest route is BAE length 12

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Exercise A, Question 2

Question:

Complete the table of least distances for the network. State the route you used for each of your entries.

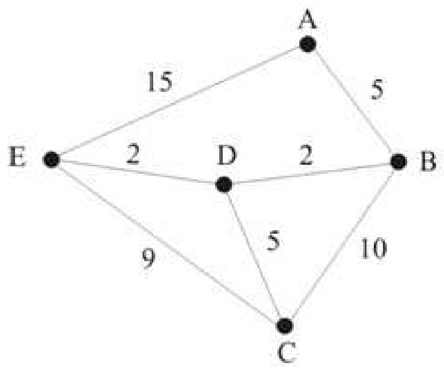

Table of least differences

	A	B	C	D	E
A	–	5		7	
B	5	–		2	
C			–	5	
D	7	2	5	–	2
E				2	–

Solution:

	A	B	C	D	E
A	–	5	**12**	7	**9**
B	5	–	7	2	**4**
C	**12**	7	–	5	7
D	7	2	5	–	2
E	**9**	**4**	7	2	–

AC – the shortest route is ABDC length 12
AE – the shortest route is ABDE length 9
BC – the shortest route is BDC length 7
BE – the shortest route is BDE length 4
CE – the shortest route is CDE length 7

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Exercise A, Question 3

Question:

Complete the table of least distances for the network. State the route you used for each of your entries.

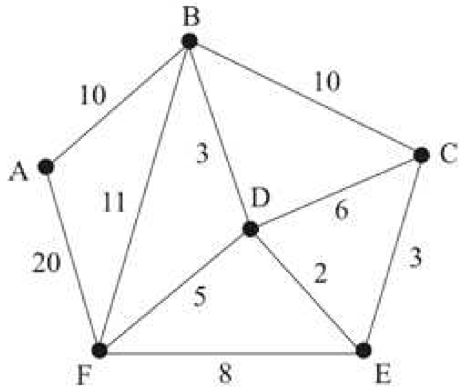

Table of least differences

	A	B	C	D	E	F
A	–	10		13	15	
B	10	–		3		
C			–		3	
D	13	3		–	2	5
E	15		3	2	–	
F				5		–

Solution:

	A	B	C	D	E	F
A	–	10	**18**	13	15	**18**
B	10	–	**8**	3	**5**	**8**
C	**18**	**8**	–	**5**	3	**10**
D	13	3	**5**	–	2	5
E	15	**5**	3	2	–	**7**
F	**18**	**8**	**10**	5	**7**	–

AC – the shortest route is ABDEC length 18
AF – the shortest route is ABDF length 18
BC – the shortest route is BDEC length 8
BE – the shortest route is BDE length 5
BF – the shortest route is BDF length 8
CD – the shortest route is CED length 5
CF – the shortest route is CEDF length 10
EF – the shortest route is EDF length 7

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Exercise A, Question 4

Question:

Complete the table of least distances for the network. State the route you used for each of your entries.

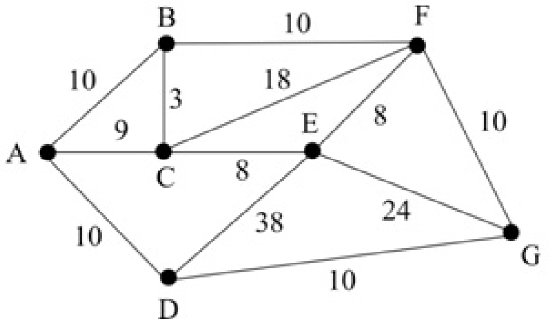

Table of least differences

	A	B	C	D	E	F	G
A	–	10	9	10	17		
B	10	–	3	20		10	20
C	9	3	–	19	8		
D	10	20	19	–		20	10
E	17		8		–	8	18
F		10		20	8	–	10
G		20		10	18	10	–

Solution:

	A	B	C	D	E	F	G
A	–	10	9	10	17	**20**	**20**
B	10	–	3	20	**11**	10	20
C	9	3	–	19	8	**13**	**23**
D	10	20	19	–	**27**	20	10
E	17	**11**	8		–	8	18
F	**20**	10	**13**	20	8	–	10
G	**20**	20	**23**	10	18	10	–

AF – the shortest route is ABF length 20
AG – the shortest route is ADG length 20
BE – the shortest route is BCE length 11
CF – the shortest route is CBF length 13
CG – the shortest route is CBFG length 23
DE – the shortest route is DACE length 27

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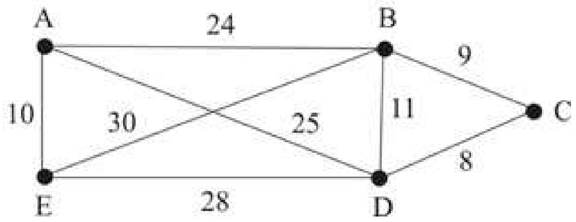
Exercise B, Question 1

Question:



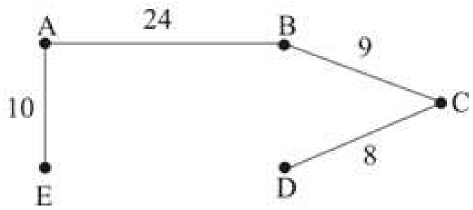
a Find a minimum spanning tree for the network above and hence find an initial upper bound for the travelling salesman problem.

b Use a shortcut to find a better upper bound

c State the route given by your improved upper bound and state its length.

Solution:

a



Initial upper bound $= 2 \times 51$
$= 102$

b Use DE as a shortcut
Route length $= 51 + 28 = 79$

c Route A B C D E A length 79

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Exercise B, Question 2

Question:

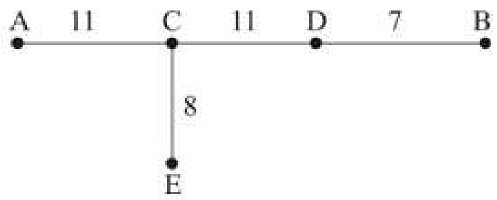
	A	B	C	D	E
A	–	13	11	19	14
B	13	–	12	7	16
C	11	12	–	11	8
D	19	7	11	–	14
E	14	16	8	14	–

A council employee needs to service five sets of traffic lights located at A, B, C, D and E. The table shows the distance, in miles between the lights. She will start and finish at A and wishes to minimise her total travelling distance.

a Find the minimum spanning tree for the network.
b Hence find an initial upper bound for the length of the employee's route.
c Use shortcuts to reduce the upper bound to a value below 65.
d State the route given by your improved upper bound and state its length.

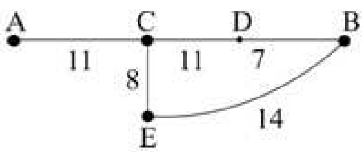
Solution:

a



b Initial upper bound $= 2 \times 37 = 74$

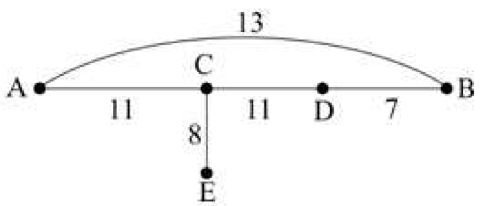
c For example **i** use BE as a shortcut



or

ii use AB as a shortcut

Other answers also possible



d i Using BE route is A C E B D C A length 62

ii Using AB route is A C E C D B A length 58

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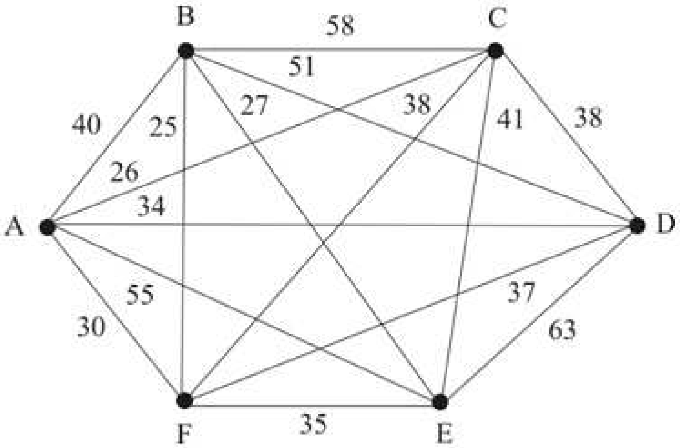
Exercise B, Question 3

Question:



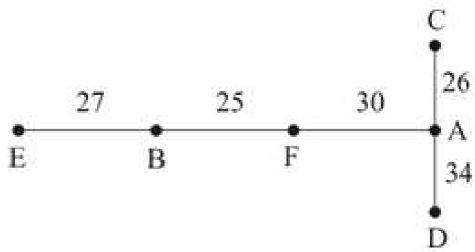
a Find a minimum spanning tree for the network above and hence find an initial upper bound for the travelling salesman problem.

b Use shortcuts to reduce the upper bound to below 240.

c State the route given by your improved upper bound and state its length.

Solution:

a



Initial upper bound $= 2 \times 142$

$= 284$

b Many possibilities: for example
DE or EC or DF and EC

c DE gives A C A F B E D A length 231
EC gives A D A F B E C A length 217
DF and EC gives A C E D F D A length 190

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Exercise B, Question 4

Question:

	S	V	W	X	Y	Z
S	–	75	30	55	70	70
V	75	–	55	30	40	15
W	30	55	–	65	45	55
X	55	30	65	–	15	10
Y	70	40	45	15	–	20
Z	70	15	55	10	20	–

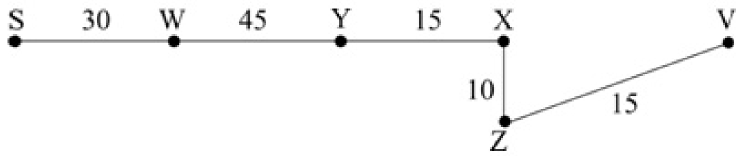
The table shows the time, in minutes, taken to travel between a surgery S and five farms V, W, X, Y and Z. A vet needs to visit animals at each of the farms and wishes to minimise the total travel time. He will start and finish at the surgery, S.

a Find a minimum spanning tree for the network above and hence find an initial upper bound for the travelling salesman problem.

b Use shortcuts to reduce the upper bound to below 200.

c State the route given by your improved upper bound and state its length.

Solution:

a 

Initial upper bound $2 \times 115 = 230$

b For example arc VS

c Route S W Y X Z V X S length 190

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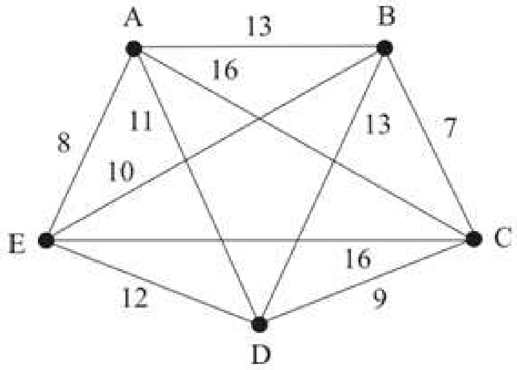
Exercise C, Question 1

Question:

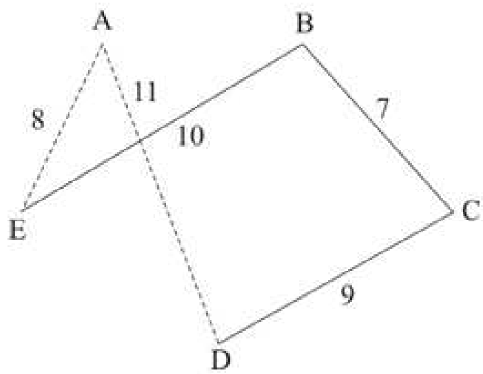


a By deleting vertex A, find a lower bound to the travelling salesman problem for the network above.

b Comment on your answer.

Solution:

a



Weight of residual minimum spanning tree = 26
Two shortest arcs from A, AE and AD
Lower bound = 26 + 8 + 11
= 45

b This is a route, so it is optimal

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Exercise C, Question 2

Question:

	A	B	C	D	E
A	–	13	11	19	14
B	13	–	12	7	16
C	11	12	–	11	8
D	19	7	11	–	14
E	14	16	8	14	–

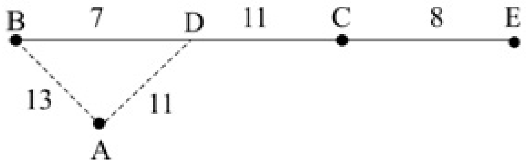
A council employee needs to service five sets of traffic lights located at A, B, C, D and E. The table shows the distance, in miles between the lights. She will start and finish at A and wishes to minimise her total travelling distance.

a By deleting vertices A then B find two lower bounds for the employee's route.

b Select the better lower bound, giving a reason for your answer.

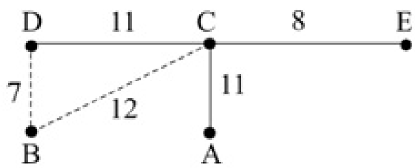
Solution:

a *Deleting A*



Weight of residual minimum spanning tree = 26
Two shortest arcs are AD and AB
Lower bound = 26 + 11 + 13
= 50

Deleting B
Weight of residual minimum spanning, tree = 30
Two shortest arcs are BD and BC
Lower bound = 30 + 7 + 12
= 49



b The better lower bound is 50 since it is higher.

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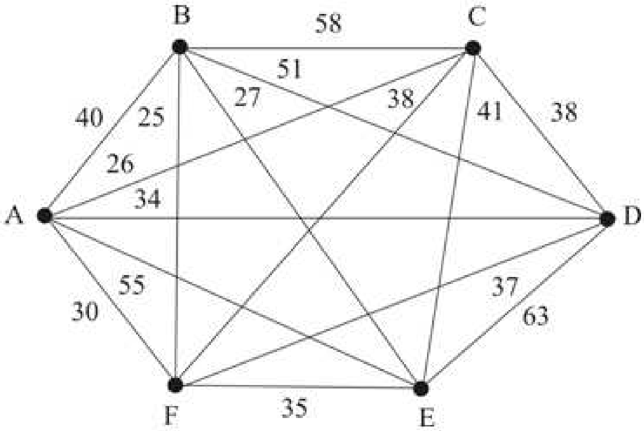
Exercise C, Question 3

Question:



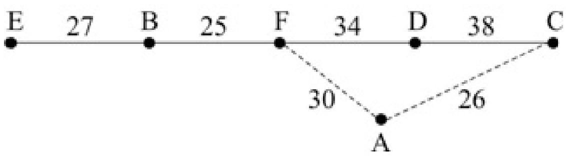
a By deleting vertices A then B, find two lower bounds for the travelling salesman problem.

b Select the better lower bound, giving a reason for your answer.

c Use inequalities, your answer to **b** and the better upper bound found in Exercise 3B Question 3, to write down the smallest interval containing the optimal route

Solution:

a *Deleting A*



Weight of residual minimum spanning tree $= 124$

Two shortest arcs are AC and AF

Lower bound $= 124 + 26 + 30 = 180$

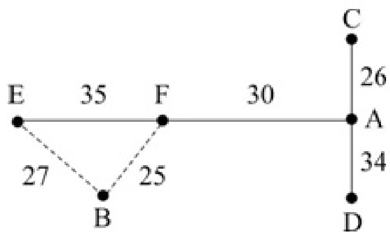
a *Deleting B*

Weight of residual minimum spanning tree $= 125$

Two least arcs BF and BE

lower bound $= 125 + 25 + 27$

$= 177$



b The better lower bound is 180 because it is higher

c $180 <$ optimal value ≤ 190 (or your answer to Exercise 3B question 3**c**)

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Exercise C, Question 4

Question:

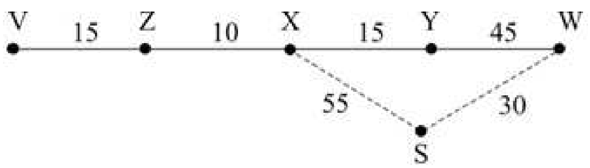
	S	V	W	X	Y	Z
S	–	75	30	55	70	70
V	75	–	55	30	40	15
W	30	55	–	65	45	55
X	55	30	65	–	15	10
Y	70	40	45	15	–	20
Z	70	15	55	10	20	–

The table shows the time, in minutes, taken to travel between a surgery S and five farms V, W, X, Y and Z. A vet needs to visit animals at each of the farms and wishes to minimise the total travel time. He will start and finish at the surgery, S.

a By deleting vertices S then V, find two lower bounds for the vet's route.
b Select the better lower bound, giving a reason for your answer.
c Use inequalities, your answer to **b** and the better upper bound found in Exercise 3B Question 4, to write down the smallest interval containing the optimal route.

Solution:

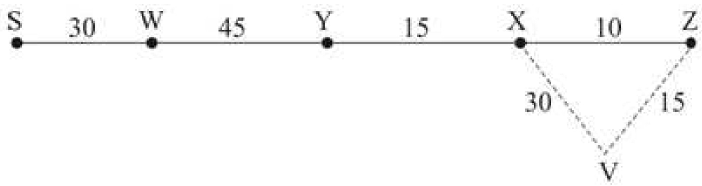
a *Deleting S*



Weight of residual minimum spanning tree = 85
Two least arcs SW and SX

Lower bound = 85 + 30 + 55
= 170

Deleting V



Weight of residual minimum spanning tree = 100
Two least arcs VZ and VX
Lower bound = 100 + 15 + 30
= 145

b The better lower bound is 170 because it is higher

c $170 < \text{optimal value} \leq 190$ (or your answer to Exercise 3B question 4**c**)

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Exercise D, Question 1

Question:

(This is the same problem as described in Exercise 3C Question 2)

	A	B	C	D	E
A	–	13	11	19	14
B	13	–	12	7	16
C	11	12	–	11	8
D	19	7	11	–	14
E	14	16	8	14	–

A council employee needs to service five sets of traffic lights located at A, B, C, D and E. The table shows the distance, in miles between the lights. She wishes to minimise her total travelling distance.

a Starting at D, find a nearest neighbour route to give an upper bound for the council employee's route.
b Show that there are two nearest neighbour routes starting from E.
c Select the value that should be given as the upper bound, give a reason for your answer.

Solution:

a $D_7B_{12}C_8E_{14}A_{19}D = 60$

b $E_8C_{11}A_{13}B_7D_{14}E = 53$
or
$E_8C_{11}D_7B_{13}A_{14}E = 53$

c The better upper bound is 53 since this is lower.

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Exercise D, Question 2

Question:

(This is the same problem as described in Exercise 3C Question 4)

	S	V	W	X	Y	Z
S	–	75	30	55	70	70
V	75	–	55	30	40	15
W	30	55	–	65	45	55
X	55	30	65	–	15	10
Y	70	40	45	15	–	20
Z	70	15	55	10	20	–

The table shows the time, in minutes, taken to travel between a surgery S and five farms V, W, X, Y and Z. A vet needs to visit animals at each of the farms and wishes to minimise the total travel time.

a Starting at Z, find a nearest neighbour route.
b Find two further nearest neighbour routes starting at X then V.
c Select the value that should be given as the upper bound, give a reason for your answer.

Solution:

a $Z_{10}X_{15}Y_{40}V_{55}W_{30}S_{70}Z = 220$

b $X_{10}Z_{15}V_{40}Y_{45}W_{30}S_{55}X = 195$

$V_{15}Z_{10}X_{15}Y_{45}W_{30}S_{75}V = 190$

c The better upper bound is 190 because it is lower

Solutionbank D2
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Exercise D, Question 3

Question:

	R	S	T	U	V	W
R	–	150	210	150	120	240
S	150	–	210	120	210	240
T	210	210	–	120	150	180
U	150	120	120	–	180	270
V	120	210	150	180	–	300
W	240	240	180	270	300	–

A printing company prints six magazines R, S, T, U, V and W, each week. The printing equipment needs to be set up differently for each magazine and the table shows the time, in minutes, needed to set up the equipment from one magazine to another. The printer must print magazine R at the start of the first day each week so the equipment is already set up to print magazine R, and must be left set up for magazine R at the end of the week. The other magazines can be printed in any order.

a If the magazines were printed in the order RSTUVWR, how long would it take in total to set up the equipment?

b Show that there are two nearest neighbour routes starting from U.

c Show that there are three nearest neighbour routes starting from V.

d Select the value that should be given as the upper bound, give a reason for your answer.

Solution:

a $R_{150}S_{210}T_{120}U_{180}V_{300}W_{240}R = 1200$ minutes

b $U_{120}S_{150}R_{120}V_{150}T_{180}W_{270}U = 990$

and

$U_{120}T_{150}V_{120}R_{150}S_{240}W_{270}U = 1050$

c $V_{120}R_{150}S_{120}U_{120}T_{180}W_{300}V = 990$

and

$V_{120}R_{150}U_{120}S_{210}T_{180}W_{300}V = 1080$

and

$V_{120}R_{150}U_{120}T_{180}W_{240}S_{210}V = 1020$

d The better upper bound is 990 because it is lower.

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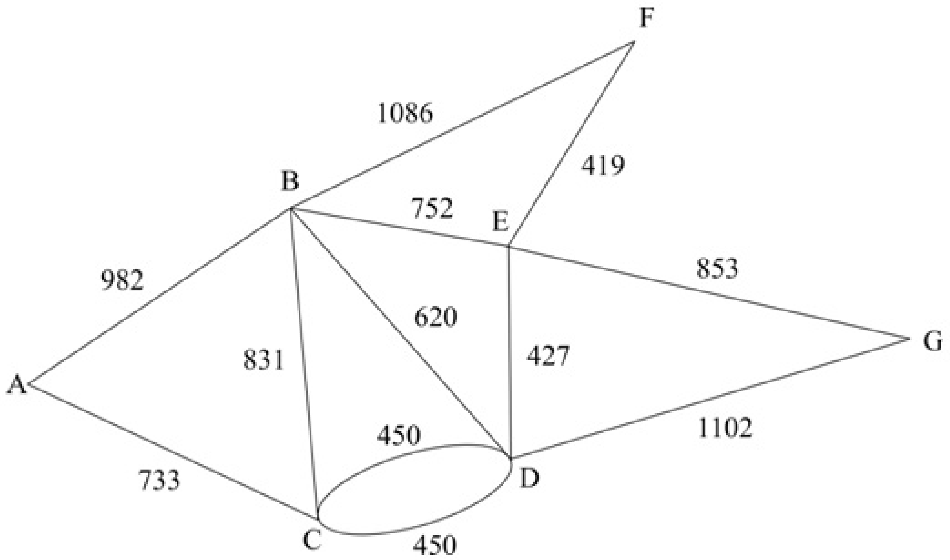
Exercise E, Question 1

Question:



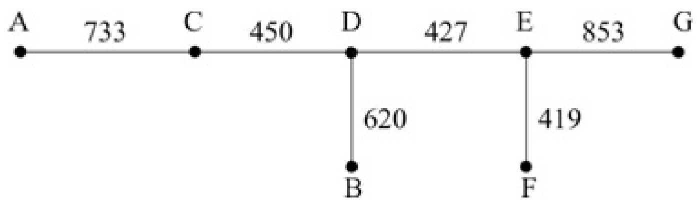
a Use an efficient algorithm to find a minimum connector for the network above. You must make your method clear.

b Hence find an initial upper bound for the travelling salesman problem.

c Use the method of short cuts to find an upper bound below 6100.

Solution:

a Either Kruskal: EF, DE, CD, BD, AC, EG or Prim (e.g.): AC, CD, DE, EF, BD, EG



b $2 \times 3502 = 7004$

c For example use AB and DG

Route A C D E F E G D B A length 6005

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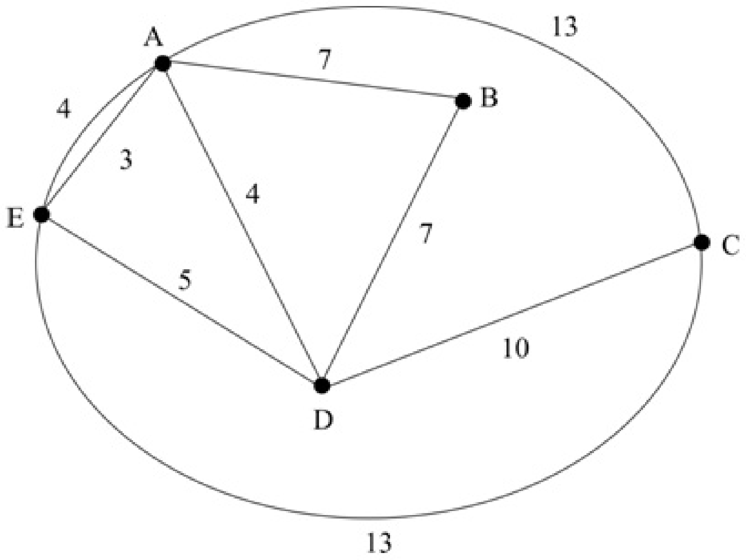
Exercise E, Question 2

Question:



The network above shows a number of hostels in a national park and the possible paths joining them. The numbers on the edges give the lengths, in km, of the paths.

a Draw a complete network showing the shortest distances between the hostels. (You may do this by inspection. The application of an algorithm is not required.)

b Use the nearest neighbour algorithm on the complete network to obtain an upper bound to the length of a tour in this network which starts and finishes at A and visits each hostel exactly once.

c Interpret your result in part **b** in terms of the original network.

Solution:

a

	A	B	C	D	E
A	–	7	13	4	3
B	7	–	17	7	10
C	13	17	–	10	13
D	4	7	10	–	5
E	3	10	13	5	–

b $A_3E_5D_7B_{17}C_{13}A = 45$

c A E D B D C A (BC is not on the original network)

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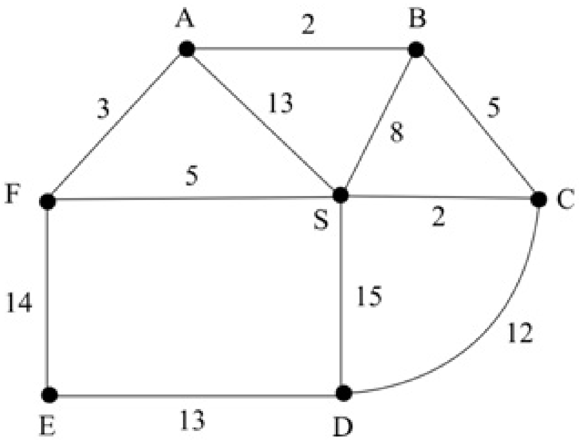
Exercise E, Question 3

Question:

(This is the network given in example 2)



The table of least distances below was formed from the network, N, above.

	S	A	B	C	D	E	F
S	–	8	7	2	14	19	5
A	8	–	2	7	19	17	3
B	7	2	–	5	17	19	5
C	2	7	5	–	12	21	7
D	14	19	17	12	–	13	19
E	19	17	19	21	13	–	14
F	5	3	5	7	19	14	–

The table shows the distances, in km, between the central sorting office at S and six post offices A, B, C, D, E and F.

A postal worker will leave the sorting office, go to each post office to collect mail and return to the sorting office. He wishes to minimise his route.

a Use Prim's algorithm, starting at S, to obtain two minimum spanning trees. State the order in which you select the arcs.
b Hence find an initial upper bound for the postal worker's route.
c Starting from this upper bound, use shortcuts to reduce the upper bound to a value below 60 km. You must state the shortcuts you use.
d Starting at C, and then at D, find two nearest neighbour routes stating their lengths.
e Select the better upper bound from your answers to **c** and **d**, give a reason for your answer.

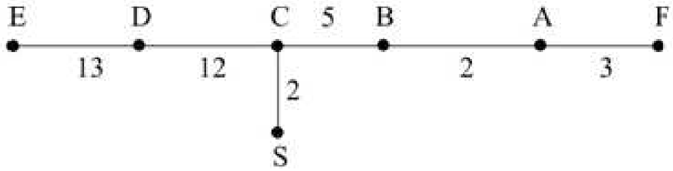
f Interpret your answer to **e** in terms of the original network, N, of roads.
g Using the table of least distances, and by deleting C, find a lower bound for the postal worker's route.

Solution:

a SC SF FA AB CD DE – tree 1

and

SC CB BA AF CD DE – tree 2



b Weight of each tree is 37
So initial upper bound is $2 \times 37 = 74$

c From tree 1
Use BE as a shortcut
(Route is S C D E B A F S) length 56

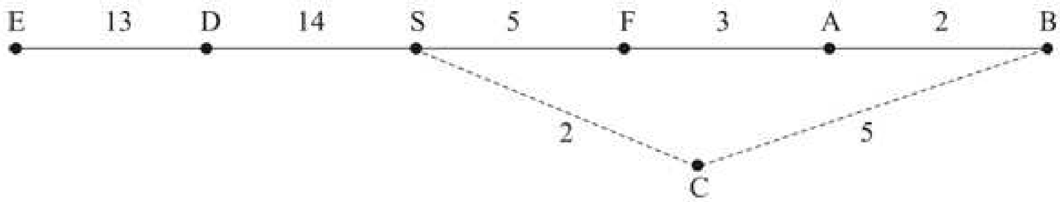
From tree 2
Use EF as a shortcut
(Route is S C B A F E D C S) length 53

d $C_2S_5F_3A_2B_{17}D_{13}E_{21}C = 63$
$D_{12}C_2S_5F_3A_2B_{19}E_{13}D = 56$

e The better upper bound is 53 since it is smaller

f The route is S C B A F E D C S

g



Weight of residual minimum spanning tree = 37
Two least arcs from C are CS and CB

Lower bound $= 37 + 2 + 5$
$= 44$

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Exercise E, Question 4

Question:

a Explain the difference between the classical and practical travelling salesman problem.

	P	Q	R	S	T	U	V
P	–	19	30	45	38	33	29
Q	19	–	28	27	50	23	55
R	30	28	–	51	29	49	50
S	45	27	51	–	77	21	71
T	38	50	29	77	–	69	37
U	33	23	49	21	69	–	56
V	29	55	50	71	37	56	–

The table shows the travel time, in minutes, between seven town halls P, Q, R, S, T, U and V. Kim works at P and must visit each of the other town halls to deliver leaflets. She wishes to minimise her route.

b Find a minimum connector for the network. You must make your method clear by listing the arcs in order of selection.

c Use the minimum connector and shortcuts to find an upper bound below 220. You must list the shortcuts you use and your final route.

d Starting at P, find a nearest neighbour route and state its length.

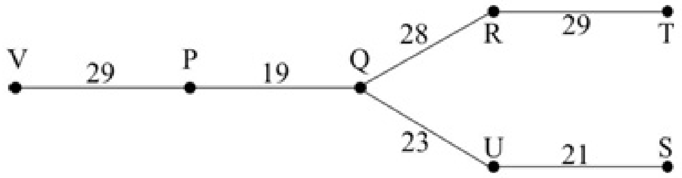
e Find a lower bound for the length of the route by deleting P.

f Looking at your answers to **c**, **d** and **e**, use inequalities to write down the smallest interval containing the optimal solution.

Solution:

a In the classical problem each vertex must be visited *exactly* once before returning to the start.
In the practical problem each vertex must be visited *at least* once before returning to the start.

b

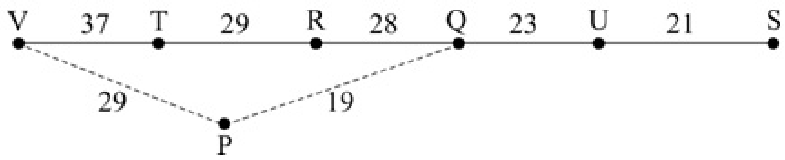


Order of arcs: PQ, QU, US, QR, $\begin{Bmatrix} \text{TR} \\ \text{VP} \end{Bmatrix}$

c Use VT and QS as shortcuts giving a length of 213
(Route P Q U S Q R T V P)

d $P_{19}Q_{22}U_{21}S_{51}R_{29}T_{37}V_{29}P = 209$

e



Weight of residual minimum spanning tree $= 140$
Two least arcs PQ and PV

$$\text{Lower bound} = 138 + 19 + 29$$
$$= 186$$

f $186 < \text{ optimal value } \leq 209$

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Exercise E, Question 5

Question:

	A	B	C	D	E	F	G
A	–	103	89	42	54	143	153
B	103	–	60	98	56	99	59
C	89	60	–	65	38	58	77
D	42	98	65	–	45	111	139
E	54	56	38	45	–	95	100
F	143	99	58	111	95	–	75
G	153	59	77	139	100	75	–

A computer supplier has outlets in seven cities A, B, C, D, E, F and G. The table shows the distances, in km, between each of these seven cities. John lives in city A and has to visit each of these cities to advise on displays. He wishes to plan a route starting and finishing at A, visiting each city and covering a minimum distance.

a Obtain a minimum spanning tree for this network explaining briefly how you applied the algorithm that you used. (Start with A and state the order in which you selected the arcs used in your tree.)

b Hence determine an initial upper bound for the length of the route travelled by John.

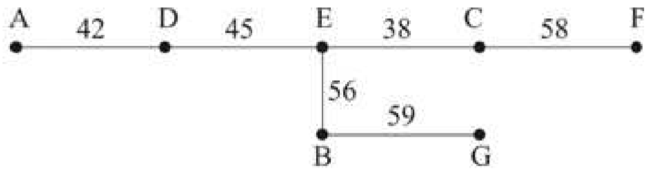
c Explain why the upper bound found in this way is unlikely to give the minimum route length.

d Starting from your initial upper bound and using an appropriate method, find an upper bound for the length of the route which is less than 430 km.

e By deleting city A, determine a lower bound for the length of John's route.

f Explain under what circumstances a lower bound obtained by this method might be an optimum solution.

Solution:

a 

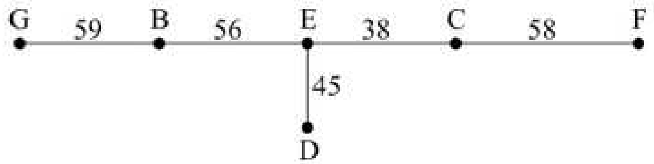
order of arcs: AD, DE, EC, EB, CF, BG

b Initial upper bound $= 2 \times 298$

$= 596$

c The minimum connector has been doubled and each arc in it repeated

d Use AE and GF as shortcuts - length 427
(route is A D E B G F C E A)

e 

Weight of residual minimum spanning tree = 256
Two least arcs from A are AD (42) and AE (54)
Lower bound = 256 + 42 + 54 = 352 km

f The lower bound will give the optimal solution if it is a tour.
If the minimum spanning tree has no 'branches' – so the two end vertices have valency 1, and all other vertices have valency 2, then if the two least arcs are incident on the 2 vertices of valency 1 an optimal solution cannot be found.

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Exercise E, Question 6

Question:

	L	C	O	B	N	E
London (L)	–	80	56	120	131	200
Cambridge (C)	80	–	100	98	87	250
Oxford (O)	56	100	–	68	103	154
Birmingham (B)	120	98	68	–	54	161
Nottingham (N)	131	87	103	54	–	209
Exeter (E)	200	250	154	161	209	–

A sales representative, Sheila, has to visit clients in six cities, London, Cambridge, Oxford, Birmingham, Nottingham and Exeter. The table shows the distances, in miles, between these six cities. Sheila lives in London and plans a route starting and finishing in London. She wishes to visit each city and drive the minimum distance.

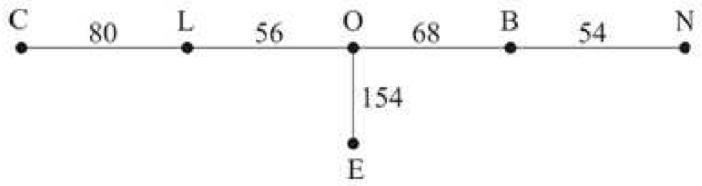
a Starting from London, use Prim's algorithm to obtain a minimum spanning tree. Show your working. State the order in which you selected the arcs and draw the tree.

b **i** Hence determine an initial upper bound for the length of the route planned by Sheila.

ii Starting from your initial upper bound and using shortcuts, obtain a route which is less than 660 miles.

c By deleting Exeter from the table determine a lower bound for the length of Sheila's route.

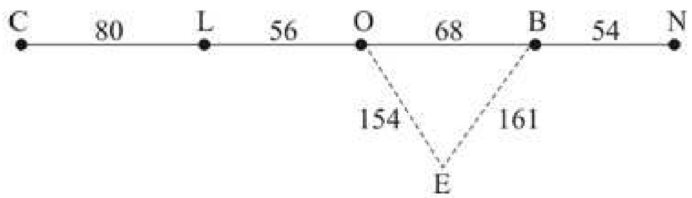
Solution:

a 

order of selection: LO, OB, BN, LC, OE

b **i** Initial upper bound $= 2 \times 412$
$= 824$ miles

ii Use NC as a shortcut – length is 653
(Route is L O E O B N C L)

c 

Weight of residual minimum spanning tree = 258
Two least arcs are EO and EB

Lower bound $= 258 + 154 + 161$
$= 573$

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Exercise A, Question 1

Question:

Formulate the following problem as a linear programming problem. You must define your variables, state your objective and write your constraints as **equations**.

A company makes three types of metal box, round, square and rectangular. Each box has to pass through two machines to be cut and formed. The round, square and rectangular boxes need 4, 2 and 3 minutes respectively on the cutter and 2, 3 and 3 on the former. Both machines are available for 6 hours per day.
The profit, in pence, made on each round, square and rectangular box is 12, 10 and 11 respectively. The company wishes to maximise its profit.

Solution:

Let x_1, x_2 and x_3 be the number of round, square and rectangular boxes respectively.

Maximise $P = 12x_1 + 10x_2 + 11x_3$

Subject to:

$$4x_1 + 2x_2 + 3x_3 + r = 360$$
$$2x_1 + 3x_2 + 3x_3 + s = 360$$
$$x_1, x_2, x_3, r, s \geq 0$$

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Exercise A, Question 2

Question:

Formulate the following problem as a linear programming problem. You must define your variables, state your objective and write your constraints as **equations**.

A company makes four different types of backpack, A, B, C and D. Each type A uses 2.5 units of material, needs 10 minutes of cutting time and 5 minutes of stitching time. These figures, together with those for types B, C and D are shown in the table

	A	B	C	D
Material in units	2.5	3	2	4
Cutting time in minutes	10	12	8	15
Stitching time in minutes	5	7	4	9

There are 1400 units of material available each week, 150 hours per week available on the cutting machine and 80 hours available on the stitching machine.
Market research says that they will sell at most 500 backpacks each week.
The profit, in pounds, is 8, 7, 6 and 9 for types A, B, C and D respectively. The company wishes to maximise its profit.

Solution:

Let x_A, x_B, x_C and x_D be the number of type A, B, C and D backpacks made

Maximise $P = 8x_A + 7x_B + 6x_C + 9x_D$

Subject to:

$$2.5x_A + 3x_B + 2x_C + 4x_D + r = 1400$$
$$10x_A + 12x_B + 8x_C + 15x_D + s = 9000$$
$$5x_A + 7x_B + 4x_C + 9x_D + t = 4800$$
$$x_A, x_B, x_C, x_D, r, s, t \geq 0$$

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Exercise A, Question 3

Question:

Formulate the following problem as a linear programming problem. You must define your variables, state your objective and write your constraints as **equations**.

The annual subscription to a bowls club is £40 for adults £10 for children and £20 for seniors.
The total number of members is restricted to 100.
At most half the club must be children and at least a third must be adults.
The club wishes to maximise its income.

Solution:

Let x_A, x_C and x_s be the number of adults, children and senior members

Maximise $P = 40x_A + 10x_C + 20x_s$

Subject to:

$$x_A + x_C + x_s + r = 100$$
$$-x_A + x_C - x_s + s = 0$$
$$-2x_A + x_C + x_s + t = 0$$
$$x_A, x_C, x_s, r, s, t \geq 0$$

$$x_C \leq \frac{1}{2}(x_A + x_C + x_s)$$
$$\frac{1}{2}x_C \leq \frac{1}{2}x_A + \frac{1}{2}x_s$$
$$x_C \leq x_A + x_s$$
$$x_C - x_A - x_s \leq 0$$

$$x_A \geq \frac{1}{3}(x_A + x_C + x_s)$$
$$3x_A \geq x_A + x_C + x_s$$
$$2x_A \geq x_C + x_s$$
$$x_C + x_s - 2x_A \leq 0$$

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Exercise A, Question 4

Question:

Formulate the following problem as a linear programming problem. You must define your variables, state your objective and write your constraints as **equations**.

Mrs Brown was rather alarmed to discover from her children at bedtime that (a week ago) they had promised she would make at least 100 small cakes for a cake sale at school the next day. Not wishing to let her children down, she puts the oven on and checks her cupboards and finds she has 3 kg of flour, 2 kg of butter and 1.5 kg of sugar, as well as other ingredients. Mrs Brown finds three cake recipes for rock cakes, fairy cakes and muffins. The recipe for rock cakes uses 220 g of flour, 100 g butter and 50 g sugar and makes 8 cakes. The recipe for fairy cakes uses 100 g each of flour, butter and sugar and makes 18 cakes. The recipe for muffins uses 250 g of flour, 50 g butter and 75 g sugar and makes 12 muffins. Each batch of rock cakes, fairy cakes and muffins take 10 minutes, 20 minutes and 15 minutes respectively to prepare.
Mrs Brown wishes to minimise her preparation time.

Solution:

Let x_r, x_f and x_m be the number of batches of rock cakes, fairy cakes and muffin made.

Minimise $T = 10x_r + 20x_f + 15x_m$

Subject to:

$$220x_r + 100x_f + 250x_m + r = 3000$$
$$100x_r + 100x_f + 50x_m + s = 2000$$
$$50x_r + 100x_f + 75x_m + t = 1500$$
$$8x_r + 18x_f + 12x_m - u = 100$$
$$x_r, x_f, x_m, r, s, t, u \geq 0$$

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Exercise A, Question 5

Question:

Formulate the following problem as a linear programming problem. You must define your variables, state your objective and write your constraints as **equations**.

Roma is moving house. She needs to pack all her extensive collection of china into special cardboard boxes which will be sold to her by the removal company. There are three sizes of box, small, medium and large. The small boxes have a capacity of 0.1 m^3 and will hold a maximum of 3 kg. The medium boxes have a capacity of 0.3 m^3 and will hold a maximum weight of 8 kg. The large boxes have a capacity of 0.7 m^3 and will hold a maximum weight of 18 kg. An expert from the removal company informs her that she allow for at least 28 m^3 packing capacity and for at last 600 kg.

Roma decides that at least half of the boxes she uses should be small and that she should use at least twice as many medium as large.

She will be able to fill the boxes she buys and the cost of each small, medium and large box is 30p, 50p and 80p.

Roma wishes to minimise the cost of the boxes she buys.

Solution:

Let x_s, x_m and x_l be the number of small, medium and large boxes.

Minimise $C = 0.3x_s + 0.5x_m + 0.8x_l$

Subject to:

$$x_s + 3x_m + 7x_l - r = 280$$
$$3x_s + 8x_m + 18x_l - s = 600$$
$$-x_s + x_m + x_l + t = 0$$
$$-x_m + 2x_l + u = 0$$
$$x_s, x_m, x_l, r, s, t, u \geq 0$$

$$x_s \geq \frac{1}{2}(x_s + x_m + x_l)$$
$$x_s \geq x_m + x_l$$
$$x_m + x_l - x_s \leq 0$$

$$x_m \geq 2x_l$$
$$2x_l - x_m \leq 0$$

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Exercise B, Question 1

Question:

Solve this linear programming problem using the simplex tableau algorithm.

Maximise $P = 5x + 6y + 4z$
Subject to

$$x + 2y + r = 6$$
$$5x + 3y + 3z + s = 24$$
$$x, y, z, r, s \geq 0$$

Solution:

b.v.	x	y	z	r	s	value	θ values
r	1	2	0	1	0	6	3 *
s	5	3	3	0	1	24	8
P	−5	−6	−4	0	0	0	

↑

b.v.	x	y	z	r	s	value	Row operations
y	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	3	R1 ÷ 2
s	$\frac{7}{2}$	0	3	$\frac{-3}{2}$	1	15	R2 − 3R1
P	−2	0	−4	3	0	18	R3 + 6R1

b.v.	x	y	z	r	s	value	Row operations
y	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	3	R1 (no change)
z	$\frac{7}{6}$	0	1	$-\frac{1}{2}$	$\frac{1}{3}$	5	R2 ÷ 3
P	$\frac{8}{3}$	0	0	1	$\frac{4}{3}$	38	R3 + 4R2

$P = 38 \quad x = 0 \quad y = 3 \quad z = 5 \quad r = 0 \quad s = 0$

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Exercise B, Question 2

Question:

Solve this linear programming problem using the simplex tableau algorithm.

Maximise $P = 3x + 4y + 10z$

Subject to

$$x + 2y + 2z + r = 100$$
$$x + 4z + s = 40$$
$$x, y, z, r, s \geq 0$$

Solution:

b.v.	x	y	z	r	s	value	θ values
r	1	2	2	1	0	100	50
s	1	0	4	0	1	40	10*
P	-3	-4	-10	0	0	0	

b.v.	x	y	z	r	s	value	Row operations
r	$\frac{1}{2}$	2	0	1	$\frac{-1}{2}$	80	R1 − 2R2
z	$\frac{1}{4}$	0	1	0	$\frac{1}{4}$	10	R2 ÷ 4
P	$\frac{-1}{2}$	-4	0	0	$\frac{5}{2}$	100	R3 + 10R2

b.v.	x	y	z	r	s	value	Row operations
y	$\frac{1}{4}$	1	0	$\frac{1}{2}$	$\frac{-1}{4}$	40	R1 ÷ 2
z	$\frac{1}{4}$	0	1	0	$\frac{1}{4}$	10	R2 (no change)
P	$\frac{1}{2}$	0	0	2	$\frac{3}{2}$	260	R3 + 4R1

$P = 260 \quad x = 0 \quad y = 40 \quad z = 10 \quad r = 0 \quad s = 0$

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Exercise B, Question 3

Question:

Solve this linear programming problem using the simplex tableau algorithm.

Maximise $P = 3x + 5y + 2z$

Subject to

$$3x + 4y + 5z + r = 10$$
$$x + 3y + 10z + s = 5$$
$$x - 2y + t = 1$$
$$x, y, z, r, s, t \geq 0$$

Solution:

b.v.	x	y	z	r	s	t	value	θ values
r	3	4	5	1	0	0	10	2.5
s	1	3	10	0	1	0	5	$1\frac{2}{3}$ *
t	1	-2	0	0	0	1	1	negative pivot
P	-3	-5	-2	0	0	0	0	

b.v.	x	y	z	r	s	t	value	Row operations
R	$\frac{5}{3}$	0	$\frac{-25}{3}$	1	$\frac{-4}{3}$	0	$\frac{10}{3}$	R1 − 4R2
Y	$\frac{1}{3}$	1	$\frac{10}{3}$	0	$\frac{1}{3}$	0	$\frac{5}{3}$	R2 ÷ 3
T	$\frac{5}{3}$	0	$\frac{20}{3}$	0	$\frac{2}{3}$	1	$\frac{13}{3}$	R3 + 2R2
P	$\frac{-4}{3}$	0	$\frac{44}{3}$	0	$\frac{5}{3}$	0	$\frac{25}{3}$	R4 + 5R2

b.v.	x	y	z	r	s	t	value	Row operations
x	1	0	-5	$\frac{3}{5}$	$\frac{-4}{5}$	0	2	R1 $\div \frac{5}{3}$
y	0	1	5	$\frac{-1}{5}$	$\frac{3}{5}$	0	1	R2 $- \frac{1}{3}$R1
t	0	0	15	-1	2	1	1	R3 $- \frac{5}{3}$R1
P	0	0	8	$\frac{4}{5}$	$\frac{3}{5}$	0	11	R4 $+ \frac{4}{3}$R1

$P = 11$ $x = 2$ $y = 1$ $z = 0$ $r = 0$ $s = 0$ $t = 1$

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Exercise B, Question 4

Question:

Solve this linear programming problem using the simplex tableau algorithm.

Maximise $P = 3x + 6y + 32z$

Subject to

$$\begin{aligned} x + 6y + 24z + r &= 672 \\ 3x + y + 24z + s &= 336 \\ x + 3y + 16z + t &= 168 \\ 2x + 3y + 32z + u &= 352 \\ x, y, z, r, s, t, u &\geq 0 \end{aligned}$$

Solution:

b.v.	x	y	z	r	s	t	u	value	θ values
r	1	6	24	1	0	0	0	672	28
s	3	1	24	0	1	0	0	336	14
t	1	3	16	0	0	1	0	168	$10\frac{1}{2}$ *
u	2	3	32	0	0	0	1	352	11
P	−3	−6	−32	0	0	0	0	0	

b.v.	x	y	z	r	s	t	u	value	row operation
r	$\frac{-1}{2}$	$\frac{3}{2}$	0	1	0	$\frac{-3}{2}$	0	420	R1 − 24R3
s	$\frac{3}{2}$	$\frac{-7}{2}$	0	0	1	$\frac{-3}{2}$	0	84	R2 − 24R3
z	$\frac{1}{16}$	$\frac{3}{16}$	1	0	0	$\frac{1}{16}$	0	$\frac{21}{2}$	R3 ÷ 16
u	0	−3	0	0	0	−2	1	16	R4 − 32R3
P	−1	0	0	0	0	2	0	336	R5 + 32R3

b.v.	x	y	z	r	s	t	u	value	Row operations
r	0	$\frac{1}{3}$	0	1	$\frac{1}{3}$	−2	0	448	$R1+\frac{1}{2}R2$
x	1	$\frac{-7}{3}$	0	0	$\frac{2}{3}$	−1	0	56	$R2\div\frac{3}{2}$
z	0	$\frac{1}{3}$	1	0	$\frac{-1}{24}$	$\frac{1}{8}$	0	7	$R3-\frac{1}{16}R2$
u	0	−3	0	0	0	−2	1	16	R4 (No change)
P	0	$\frac{-7}{3}$	0	0	$\frac{2}{3}$	1	0	392	R5 + R2

b.v.	x	y	z	r	s	t	u	value	row operations
r	0	0	−1	1	$\frac{3}{8}$	$\frac{-17}{8}$	0	441	$R1-\frac{1}{3}R3$
x	1	0	7	0	$\frac{3}{8}$	$\frac{-1}{8}$	0	105	$R2+\frac{7}{3}R3$
y	0	1	3	0	$\frac{-1}{8}$	$\frac{3}{8}$	0	21	$R3\div\frac{1}{3}$
u	0	0	9	0	$\frac{-3}{8}$	$\frac{-7}{8}$	1	79	R4 + 3R3
P	0	0	7	0	$\frac{3}{8}$	$\frac{15}{8}$	0	441	$R5+\frac{7}{3}R3$

$P=441 \quad x=105 \quad y=21 \quad z=0 \quad r=441 \quad s=0 \quad t=0 \quad u=79$

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Exercise B, Question 5

Question:

Solve this linear programming problem using the simplex tableau algorithm.

Maximise $P = 4x_1 + 3x_2 + 2x_3 + 3x_4$

Subject to

$$x_1 + 4x_2 + 3x_3 + x_4 + r = 95$$
$$2x_1 + x_2 + 2x_3 + 3x_4 + s = 67$$
$$x_1 + 3x_2 + 2x_3 + 2x_4 + t = 75$$
$$3x_1 + 2x_2 + x_3 + 2x_4 + u = 72$$
$$x_1, x_2, x_3, x_4, r, s, t, u \geq 0$$

Solution:

b.v.	x_1	x_2	x_3	x_4	r	s	t	u	value	θ values
r	1	4	3	1	1	0	0	0	95	95
s	2	1	2	3	0	1	0	0	67	63.5
t	1	3	2	2	0	0	1	0	75	75
u	3	2	1	2	0	0	0	1	72	24
P	−4	−3	−2	−3	0	0	0	0	0	

b.v.	x_1	x_2	x_3	x_4	r	s	t	u	value	Row operation
r	0	$\frac{10}{3}$	$\frac{8}{3}$	$\frac{1}{3}$	1	0	0	$\frac{-1}{3}$	71	R1 − R4
s	0	$\frac{-1}{3}$	$\frac{4}{3}$	$\frac{5}{3}$	0	1	0	$\frac{-2}{3}$	19	R2 − 2R4
t	0	$\frac{7}{3}$	$\frac{5}{3}$	$\frac{4}{3}$	0	0	1	$\frac{-1}{3}$	51	R3 − R4
x_1	1	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	0	0	0	$\frac{1}{3}$	24	R4 ÷ 3
P	0	$\frac{-1}{3}$	$\frac{-2}{3}$	$\frac{-1}{3}$	0	0	0	$\frac{4}{3}$	96	R5 + 4R4

b.v.	x_1	x_2	x_3	x_4	r	s	t	u	value	Row operation
r	0	4	0	−3	1	−2	0	1	33	$R1-\frac{8}{3}R2$
x_3	0	$\frac{-1}{4}$	1	$\frac{5}{4}$	0	$\frac{3}{4}$	0	$\frac{-1}{2}$	$\frac{57}{4}$	$R2\div\frac{4}{3}$
t	0	$\frac{11}{4}$	0	$\frac{-3}{4}$	0	$\frac{-5}{4}$	1	$\frac{1}{2}$	$\frac{109}{4}$	$R3-\frac{5}{3}R2$
x_1	1	$\frac{3}{4}$	0	$\frac{1}{4}$	0	$\frac{-1}{4}$	0	$\frac{1}{2}$	$\frac{77}{4}$	$R4-\frac{1}{3}R2$
P	0	$\frac{-1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1	$\frac{211}{2}$	$R5+\frac{2}{3}R2$

b.v.	x_1	x_2	x_3	x_4	r	s	t	u	value	Row operation
x_2	0	1	0	$\frac{-3}{4}$	$\frac{1}{4}$	$\frac{-1}{2}$	0	$\frac{1}{4}$	$\frac{33}{4}$	R1 ÷ 4
x_3	0	0	1	$\frac{17}{16}$	$\frac{1}{16}$	$\frac{5}{8}$	0	$\frac{-7}{16}$	$\frac{261}{16}$	$R2+\frac{1}{4}R1$
t	0	0	0	$\frac{21}{16}$	$\frac{-11}{16}$	$\frac{1}{8}$	1	$\frac{-3}{16}$	$\frac{73}{16}$	$R3-\frac{11}{4}R1$
x_1	1	0	0	$\frac{13}{16}$	$\frac{-3}{16}$	$\frac{1}{8}$	0	$\frac{5}{16}$	$\frac{209}{16}$	$R4-\frac{3}{4}R1$
P	0	0	0	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	0	$\frac{9}{8}$	$\frac{877}{8}$	$R5+\frac{1}{2}R1$

$P=\frac{877}{8}\quad x_1=\frac{209}{16}\quad x_2=\frac{33}{4}\quad x_3=\frac{261}{16}\quad x_4=0\quad r=0\quad s=0\quad t=\frac{73}{16}\quad u=0$

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Exercise B, Question 6

Question:

For each of the above questions 1 to 5:

a verify, using the original equations, that your solution is feasible,

b write down the final set of equations given by your optimal tableau,

c use the profit equation, written in part **b**, to explain why your solution is optimal.

Solution:

For Q1

a

$$P = 5x + 6y + 42 \qquad \mathrm{P} = 5(0) + 6(3) + 4(5) = 38$$
$$x + 2y + r = 6 \qquad 0 + 2(3) + 0 = 6$$
$$5x + 3y + 32 + 5 = 24 \qquad 5(0) + 3(3) + 3(5) + 0 = 24$$

b

$$P + 12x + r + \frac{4}{3}s = 38$$
$$\frac{1}{2}x + y + \frac{1}{2}r = 3$$
$$\frac{7}{6}x + z - \frac{1}{2}r + \frac{1}{3}s = 5$$

c $P = 38 - 12x - r - \frac{4}{3}s$ so increasing x, r or s would decrease P.

For Q2

a

$$P = 3x + 4y + 10z \Rightarrow 3(0) + 4(10) + 10(10) = 260$$
$$x + 2y + 2z + r = 100 \Rightarrow 0 + 2(40) + 2(10) + 0 = 100$$
$$x + 4z + s = 40 \Rightarrow 0 + 4(10) + 0 = 40$$

b

$$P + \frac{1}{2}x + 2r + \frac{3}{2}s = 260$$
$$\frac{1}{4}x + y + \frac{1}{2}r - \frac{1}{4}s = 40$$
$$\frac{1}{4}x + z + \frac{1}{4}s = 10$$

c $P = 260 - \frac{1}{2}x - 2r - \frac{3}{2}s$, so increasing x, r, or s would decrease P.

For Q3

a

$$P = 3x + 5y + 2z \Rightarrow 3(2) + 5(1) + 2(0) = 11$$
$$3x + 4y + 5z + r = 10 \Rightarrow 3(2) + 4(1) + 5(0) + 0 = 10$$
$$x + 3y + 10z + s = 5 \Rightarrow 2 + 3(1) + 10(0) + 0 = 5$$
$$x - 2y + t = 1 \Rightarrow 2 - 2(1) + 1 = 1$$

b

$$\mathrm{P} + 8z + \frac{4}{5}r + \frac{3}{5}s = 11$$
$$x - 5z + \frac{3}{5}r - \frac{4}{5}s = 2$$
$$y + 5z - \frac{1}{5}r + \frac{3}{5}s = 1$$
$$15z - r + 2s + t = 1$$

c $P = 11 - 8z - \frac{4}{5}r - \frac{3}{5}s$, so increasing z, r or s would decrease P.

For Q4

a $P = 3x + 6y + 32z \Rightarrow 3(105) + 6(21) + 32(0) = 441$

$x + 6y + 24z + r = 672 \Rightarrow 105 + 6(21) + 24(0) + 441 = 672$

$3x + y + 24z + s = 336 \Rightarrow 3(105) + 21 + 24(0) + 0 = 336$

$x + 3y + 16z + t = 168 \Rightarrow 105 + 3(21) + 16(0) + 0 = 168$

$2x + 3y + 32z + u = 352 \Rightarrow 2(105) + 3(21) + 32(0) + 79 = 352$

b $P + 7z + \frac{3}{8}s + \frac{15}{8}t = 441$

$-z + r + \frac{3}{8}s - \frac{17}{8}t = 441$

$x + 7z + \frac{3}{8}s - \frac{1}{8}t = 105$

$y + 3z - \frac{1}{8}s + \frac{3}{8}t = 21$

$9z - \frac{3}{8}s - \frac{7}{8}t + u = 79$

c $P = 441 - 7z - \frac{3}{8}s - \frac{15}{8}t$, so increasing z, s or t would decrease P.

For Q5

a $P = 4x_1 + 3x_2 + 2x_3 + 3x_4 \Rightarrow 4\left(\frac{209}{16}\right) + 3\left(\frac{33}{4}\right) + 2\left(\frac{261}{16}\right) + 3(0) = \frac{877}{8}$

$x_1 + 4x_2 + 3x_3 + x_4 + r = 95 \Rightarrow \frac{209}{16} + 4\left(\frac{33}{4}\right) + 3\left(\frac{261}{16}\right) + (0) + 0 = 95$

$2x_1 + x_2 + 2x_3 + 3x_4 + s = 67 \Rightarrow 2\left(\frac{209}{16}\right) + \left(\frac{33}{4}\right) + 2\left(\frac{261}{16}\right) + 3(0) + 0 = 67$

$x_1 + 3x_2 + 2x_3 + 2x_4 + t = 75 \Rightarrow \frac{209}{16} + 3\left(\frac{33}{4}\right) + 2\left(\frac{261}{16}\right) + 2(0) + \frac{73}{16} = 75$

$3x_1 + 2x_2 + x_3 + 2x_4 + u = 72 \Rightarrow 3\left(\frac{209}{16}\right) + 2\left(\frac{33}{4}\right) + \left(\frac{261}{16}\right) + 2(0) + 0 = 72$

b $P + \frac{1}{8}x_4 + \frac{1}{8}r + \frac{1}{4}s + \frac{9}{8}u = \frac{877}{8}$

$x_2 - \frac{3}{4}x_4 + \frac{1}{4}r - \frac{1}{2}s + \frac{1}{4}u = \frac{33}{4}$

$x_3 + \frac{17}{16}x_4 + \frac{1}{16}r + \frac{5}{8}s - \frac{7}{16}u = \frac{261}{16}$

$\frac{21}{16}x_4 - \frac{11}{16}r + \frac{1}{8}s + t - \frac{41}{48}u = \frac{73}{16}$

$x_1 + \frac{13}{16}x_4 - \frac{3}{16}r + \frac{1}{8}s + \frac{5}{16}u = \frac{209}{16}$

c $P = \frac{877}{8} - \frac{1}{8}x_4 - \frac{1}{8}r - \frac{1}{4}s - \frac{9}{8}u$, so increasing, x_4, r, s or u would decrease P.

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Exercise C, Question 1

Question:

In a particular factory 3 types of product, A, B and C, are made. The number of each of the products made is x, y and z respectively and P is the profit in pounds. There are two machines involved in making the products which have only a limited time available. These time limitations produce two constraints.
In the process of using the simplex algorithm the following tableau is obtained, where r and s are slack variables.

Basic variable	x	y	z	r	s	Value
z	$\frac{1}{3}$	0	1	−8	1	75
y	$\frac{2}{11}$	1	0	$\frac{17}{11}$	0	56
P	$\frac{3}{2}$	0	0	$\frac{3}{4}$	0	840

a Give one reason why this tableau can be seen to be optimal (final).

b By writing out the profit equation, or otherwise, explain why a further increase in profit is not possible under these constraints.

c From this tableau deduce

i the maximum profit,

ii the optimum number of type A, B and C that should be produced to maximise the profit.

Solution:

a There are no negative numbers in the profit row.

b $P+\frac{3}{2}x+\frac{3}{4}r=840$

So $P=840-\frac{3}{2}x-\frac{3}{4}r$

Increasing x or r would decrease P.

c **i** Maximum profit = £840

ii Optimum number of A = 0, B = 56 and C = 75

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Exercise C, Question 2

Question:

A sweet manufacturer produces packets of orange and lemon flavoured sweets.
The manufacturer can produce up to 25 000 orange sweets and up to 36 000 lemon sweets per day.

Small packets contain 5 orange and 5 lemon sweets.
Medium packets contain 8 orange and 6 lemon sweets.
Large packets contain 10 orange and 15 lemon sweets.

The manufacturer makes a profit of 14p, 20p and 30p on each of the small, medium and large packets respectively. He wishes to maximise his total daily profit.

Use x, y and z to represent the number of small, medium and large packets respectively, produced each day.

a Formulate this information as a linear programming problem, making your objective function and constraints clear. Change any inequalities to equations using r and s as slack variables.

The tableau below is obtained after one complete iteration of the simplex algorithm.

Basic variable	x	y	z	r	s	Value
r	$1\frac{2}{3}$	4	0	1	$-\frac{2}{3}$	1000
z	$\frac{1}{3}$	$\frac{2}{5}$	1	0	$\frac{1}{15}$	2400
P	−4	−8	0	0	2	72 000

b Start from this tableau and continue the simplex algorithm by increasing y, until you have either completed two complete iterations or found an optimal solution.

From your final tableau:

c **i** write down the numbers of small, medium and large packets indicated,
ii write down the profit,
iii state whether this is an optimal solution, giving your reason. **[*E*]**

Solution:

a Maximise $P = 14x + 20y + 30z$
Subject to:
$5x + 8y + 10z + r = 25\,000$
$5x + 6y + 15z + s = 36\,000$
where r and s are slack variables $x, y, z, r, s \geq 0$

b

b.v.	x	y	z	r	s	value
r	$1\frac{2}{3}$	4	0	1	$-\frac{2}{3}$	1000
z	$\frac{1}{3}$	$\frac{2}{5}$	1	0	$\frac{1}{15}$	2400
P	-4	-8	0	0	2	72 000

b.v.	x	y	z	r	s	value	Row operation
y	$\frac{5}{12}$	1	0	$\frac{1}{4}$	$-\frac{1}{6}$	250	$R1 \div 4$
z	$\frac{1}{6}$	0	1	$-\frac{1}{10}$	$\frac{2}{15}$	23 00	$R2 - \frac{2}{5}R1$
P	$-\frac{2}{3}$	0	0	2	$\frac{2}{3}$	74 000	$R3 + 8R1$

b.v.	x	y	z	r	s	Value	Row operations
x	1	$2\frac{2}{5}$	0	$\frac{3}{5}$	$-\frac{2}{5}$	600	$R1 \div \frac{5}{12}$
z	0	$-\frac{2}{5}$	1	$-\frac{1}{5}$	$\frac{1}{5}$	2200	$R2 - \frac{1}{6}R1$
p	0	$1\frac{3}{5}$	0	$2\frac{2}{5}$	$\frac{2}{5}$	74 400	$R3 + \frac{2}{3}R1$

c **i** $x = 600\ y = 0\ z = 2200$

ii Profit is $=$ £ 744

iii The solution is optimal since there are no negative numbers in the profit row.

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Exercise C, Question 3

Question:

Tables are to be bought for a new restaurant. The owners may buy small, medium and large tables that seat 2, 4 and 6 people respectively.
The owners require at most 20% of the total number of tables to be medium sized.
The tables cost £60, £100 and £160 respectively for small, medium and large. The owners have a budget of £2000 for buying tables.
Let the number of small, medium and large tables be x, y and z respectively.

a Write down 5 inequalities implied by the constraints. Simplify these where appropriate.
The owners wish to maximise the total seating capacity, S, of the restaurant.
b Write down the objective function for S in terms of x, y and z.
c Explain why it is not appropriate to use a graphical method to solve this problem.
It is decided to use the simplex algorithm to solve this problem.
d Show that a possible initial tableau is

Basic variable	x	y	z	r	t	Value
r	−1	4	−1	1	0	0
t	3	5	8	0	1	100
S	−2	−4	−6	0	0	0

It is decided to increase z first.

e Show that, after one complete iteration, the tableau becomes

Basic variable	x	y	z	r	t	Value
r	$-\frac{5}{8}$	$\frac{37}{8}$	0	1	$\frac{1}{8}$	$\frac{25}{2}$
t	$\frac{3}{8}$	$\frac{5}{8}$	1	0	$\frac{1}{8}$	$\frac{25}{2}$
S	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{3}{4}$	75

f Perform one further complete iteration.
g Explain how you can decide if your tableau is now final.
h Find the number of each type of table the restaurant should buy and their total cost. **[*E*]**

Solution:

a $\frac{1}{5}(x+y+z) \geq y \Rightarrow -x+4y-z \leq 0$

$60x+100y+160z \leq 2000 \Rightarrow 3x+5y+8z \leq 100$

$x \geq 0 \quad y \geq 0 \quad z \geq 0$

b $S = 2x+4y+6z$

c There are three variables.

d

b.v.	x	y	z	r	t	value
r	-1	4	-1	1	0	0
t	3	5	8	0	1	100
S	-2	-4	-6	0	0	0

e

b.v.	x	y	z	r	t	value	Row operations
r	$\frac{-5}{8}$	$4\frac{5}{8}$	0	1	$\frac{1}{8}$	$12\frac{1}{2}$	R1+R2
z	$\frac{3}{8}$	$\frac{5}{8}$	1	0	$\frac{1}{8}$	$12\frac{1}{2}$	R2 ÷ 8
S	$\frac{1}{4}$	$-\frac{1}{4}$	0	0	$\frac{3}{4}$	75	R3+6R2

f

b.v.	x	y	z	r	t	value	Row operations
y	$-\frac{5}{37}$	1	0	$\frac{8}{37}$	$\frac{1}{37}$	$2\frac{26}{37}$	$\text{R1} \div 4\frac{5}{8}$
z	$\frac{17}{37}$	0	1	$\frac{-5}{37}$	$\frac{4}{37}$	$10\frac{30}{37}$	$\text{R2} - \frac{5}{8}\text{R1}$
S	$\frac{8}{37}$	0	0	$\frac{2}{37}$	$\frac{28}{37}$	$75\frac{25}{37}$	$\text{R3} + \frac{1}{4}\text{R1}$

g There are no negative numbers in the objective row.

h 0 small, 2 medium and 11 large tables (seating 74) at a cost of £1960

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Exercise C, Question 4

Question:

Kuddly Pals Co. Ltd. make two types of soft toy: bears and cats. The quantity of material needed and the time taken to make each type of toy is given in the table.

Toy	Material (m^2)	Time (minutes)
Bear	0.05	12
Cat	0.08	8

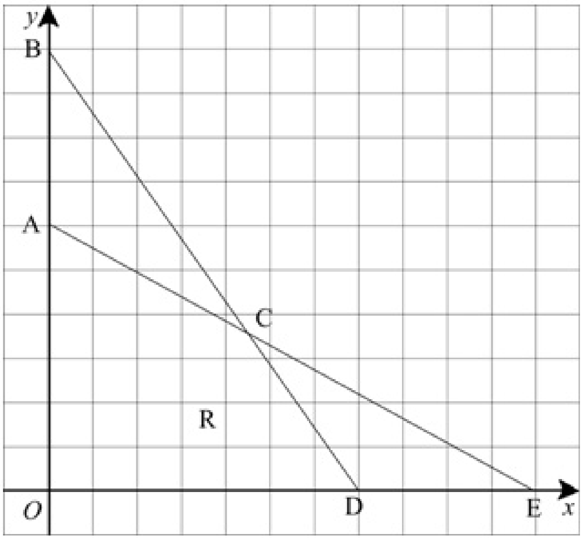
Each day the company can process up to $20\,m^2$ of material and there are 48 worker hours available to assemble the toys.
Let x be the number of bears made and y the number of cats made each day.

a Show that this situation can be modelled by the inequalities
$5x+8y \le 2000$,
$3x+2y \le 720$,
in addition to $x \ge 0, y \ge 0$.

The profit made on each bear is £1.50 and on each cat £1.75. Kuddly Pals Co. Ltd. wishes to maximise its daily profit.

b Set up an initial simplex tableau for this problem.
c Solve the problem using the simplex algorithm.

The diagram shows a graphical representation of the feasible region.



d Relate each stage of the simplex tableau to the corresponding point in the diagram. **[*E*]**

Solution:

a *Material*

(×100) $0.05x + 0.08y \le 20$

$5x + 8y \le 2000$

Time (÷4) $12x + 8y \le 2880$

$3x + 2y \le 720$

b

b.v.	x	y	r	s	value
r	5	8	1	0	2000
s	3	2	0	1	720
P	−1.5	−1.75	0	0	0

c

b.v.	x	y	r	s	value	Row operations
Y	$\frac{5}{8}$	1	$\frac{1}{8}$	0	250	$R1 \div 8$
s	$1\frac{3}{4}$	0	$-\frac{1}{4}$	1	220	R2
P	$\frac{-13}{32}$	0	$\frac{7}{32}$	0	$437\frac{1}{2}$	$R3 - 1\frac{3}{4}R1$

b.v.	x	y	r	s	value	Row operations
y	0	1	$\frac{3}{14}$	$-\frac{5}{14}$	$171\frac{3}{7}$	$R1 - \frac{5}{8}R2$
x	1	0	$-\frac{1}{7}$	$\frac{4}{7}$	$125\frac{5}{7}$	$R2 \div 1\frac{3}{4}$
P	0	0	$\frac{9}{56}$	$\frac{13}{56}$	$488\frac{4}{7}$	$R3 \div 1\frac{13}{32}R2$

c Optimal solution $x = 125\frac{5}{7}$ $y = 171\frac{3}{7}$

Integer solutions needed, so point testing gives $x = 126$ $y = 171$

d The first point is A if y is increased first
(D if x is increased first),
The second point is C.

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Exercise C, Question 5

Question:

A clocksmith makes three types of luxury wristwatch. The mechanism for each watch is assembled by hand by a skilled watchmaker and then the complete watch is formed, weatherproofed and packaged for sale by a fitter.
The table shows the times, in minutes, for each stage of the process.

Watch type	Watchmaker	Fitter
A	54	60
B	72	36
C	36	48

The watchmaker works for a maximum of 30 hours per week and the fitter for a maximum of 25 hours per week.
Let the number of type A, B and C watches made per week be x, y and z.

a Show that the above information leads to the two inequalities

$$3x+4y+2z \le 100,$$
$$5x+3y+4z \le 125.$$

The profit made on type A, B and C watches is £12, £24 and £20 respectively.

b Write down an expression for the profit, P, in pounds, in terms of x, y and z.
The clocksmith wishes to maximise his weekly profit. It is decided to use the simplex algorithm to solve this problem.
c Write down the initial tableau using r and s as the slack variables.
d Increasing y first, show that after two complete iterations of the simplex algorithm the tableau becomes

Basic variable	x	y	z	r	s	Value
y	$\frac{1}{5}$	1	0	$\frac{2}{5}$	$-\frac{1}{5}$	15
z	$\frac{11}{10}$	0	1	$-\frac{3}{10}$	$\frac{2}{5}$	20
P	$\frac{74}{5}$	0	0	$\frac{18}{5}$	$\frac{16}{5}$	760

e Give a reason why this tableau is optimal (final).
f Write down the numbers of each type of watch that should be made to maximise the profit. State the maximum profit. **[*E*]**

Solution:

a Watchmaker

$$54x + 72y + 36z \leq 1800$$

$(\div 418)$

$$3x + 4y + 2z \leq 100$$

Fitter

$$60x + 36y + 48z \leq 1500$$

$(\div 12)$

$$5x + 3y + 4z \leq 125$$

b $P = 12x + 24y + 20z$

c

b.v.	x	y	z	r	s	value
r	3	4	2	1	0	100
s	5	3	4	0	1	125
P	−12	−24	−20	0	0	0

d

b.v.	x	y	z	r	s	value	Row operations
y	$\frac{3}{4}$	1	$\frac{1}{2}$	$\frac{1}{4}$	0	25	R1 ÷ 4
s	$2\frac{3}{4}$	0	$2\frac{1}{2}$	$\frac{-3}{4}$	1	50	R2 − 3R1
P	6	0	−8	6	0	600	R3 + 24R1

b.v.	x	y	z	r	s	value	Raw operations
y	$\frac{1}{5}$	1	0	$\frac{2}{5}$	$-\frac{1}{5}$	15	R1 − $\frac{1}{2}$R2
z	$\frac{11}{10}$	0	1	$-\frac{3}{10}$	$\frac{2}{5}$	20	R2 ÷ $2\frac{1}{2}$
P	$14\frac{4}{5}$	0	0	$3\frac{3}{5}$	$3\frac{1}{5}$	760	R3 + 8R2

e There are no negative numbers in the profit row

f Type A = 0 Type B = 15 Type C = 20

Profit = £760

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Exercise C, Question 6

Question:

A craftworker makes three types of wooden animals for sale in wildlife parks. Each animal has to be carved and then sanded.
Each Lion takes 2 hours to carve and 25 minutes to sand.
Each Giraffe takes $2\frac{1}{2}$ hours to carve and 20 minutes to sand.
Each Elephant takes $1\frac{1}{2}$ hours to carve and 30 minutes to sand.
Each day the craftworker wishes to spend at most 3 hours carving and at most 2 hours sanding.
Let x be the number of Lions, y the number of Giraffes and z the number of Elephants he produces each day.
The craftworker makes a profit of £14 on each Lion, £12 on each Giraffe and £13 on each Elephant. He wishes to maximise his profit, P.

a Model this as a linear programming problem, simplifying your expressions so that they have integer coefficients.

It is decided to use the simplex algorithm to solve this problem.

b Explaining the purpose of r and s, show that the initial tableau can be written as:

Basic variable	x	y	z	r	s	Value
r	4	5	3	1	0	16
t	5	4	6	0	1	24
P	−14	−12	−13	0	0	0

c Choosing to increase x first, work out the next complete tableau, where the x column includes two zeros.

d Explain what this first iteration means in practical terms. **[*E*]**

Solution:

a Maximise $P = 14x + 12y + 13z$

Subject to:

Carving $2x + 2.5y + 1.5z \le 8 \Rightarrow 4x + 5y + 3z \le 16$

Sanding $25x + 20y + 30z \le 120 \Rightarrow 5x + 4y + 6z \le 24$

$x, y, z \ge 0$

b r and s are numbers which indicate the slack time

Profit: $P - 14x - 12y - 13z = 0$

Constraints:

$4x + 5y + 3z + r = 16$

$5x + 4y + 6z + s = 24$

b.v.	x	y	z	r	s	value
r	4	5	3	1	0	16
s	5	4	6	0	1	24
P	-14	-12	-13	0	0	0

c

b.v.	x	y	z	r	s	value	Row operations
x	1	$\frac{5}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	0	4	$R1 \div 4$
s	0	$\frac{-9}{4}$	$\frac{9}{4}$	$\frac{-5}{4}$	1	4	$R2 - 5R1$
P	0	$\frac{11}{2}$	$\frac{-5}{2}$	$\frac{7}{2}$	0	56	$R3 + 14R1$

d From a zero stock situation, if we increase the number of lions to 4, we are increasing the profit from 0 to £56.

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 1

Question:

A two person zero-sum game is represented by the following pay-off matrix for player A.

	B plays 1	B plays 2	B plays 3
A plays 1	3	2	3
A plays 2	−2	1	3
A plays 3	4	2	1

a Determine the play safe strategy for each player.
b Verify that there is a stable solution for this game and determine the saddle point.

Solution:

a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	3	2	3	2	$\leftarrow$
A plays 2	−2	1	3	−2	
A plays 3	4	2	1	1	
Column max	4	2	3		
		$\uparrow$			

A should play 1 (row maximin = 2)
B should play 2 (column minimax = 2)

b row maximin = 2 = column minimax
$\therefore$ game is stable

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Exercise A, Question 2

Question:

Robert and Steve play a zero-sum game. This game is represented by the following pay-off matrix for Robert.

	Steve plays 1	Steve plays 2	Steve plays 3	Steve plays 4
Robert plays 1	-2	-1	-3	1
Robert plays 2	2	3	1	-2
Robert plays 3	1	1	-1	3

a Determine the play safe strategy for each player.
b Verify that there is no stable solution for this game.

Solution:

a

	S plays 1	S plays 2	S plays 3	S plays 4	Row min	
R plays 1	-2	-1	-3	1	-3	
R plays 2	2	3	1	-2	-2	
R plays 3	1	1	-1	3	-1	$\leftarrow$
Column max	2	3	1	3		
			$\uparrow$			

R should play 3 (row maximin $= -1$)
S should play 3 (column minimax $= 1$)

b row maximin $\neq$ column minimax

$$-1 \neq 1$$

so game is not stable

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Exercise A, Question 3

Question:

A two person zero-sum game is represented by the following pay-off matrix for player A.

	B plays 1	B plays 2	B plays 3
A plays 1	−3	−2	2
A plays 2	−1	−1	3
A plays 3	4	−3	1
A plays 4	3	−1	−1

a Determine the play safe strategy for each player.

b Verify that there is a stable solution for this game and determine the saddle points.

c State the value of the game to player A.

Solution:

a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	−3	−2	2	−3	
A plays 2	−1	−1	3	−1	←
A plays 3	4	−3	1	−3	
A plays 4	3	−1	−1	−1	←
Column max	4	−1	3		
		↑			

A should play 2 or 4 (row maximin −1)
B should play 2 (column minimax −1)

b Since row maximin = column minimax

$$-1 = -1$$

game is stable

Saddle points are (A2, B2) and (A4, B2).

c Value of the game is −1 to A (if A players 2 or 4 and B plays 2 the value of the game is −1).

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Exercise A, Question 4

Question:

Claire and David play a two person zero-sum game, which is represented by the following pay-off matrix for Claire.

	D plays 1	D plays 2	D plays 3	D plays 4
C plays 1	7	2	−3	5
C plays 2	4	−1	1	3
C plays 3	−2	5	2	−1
C plays 4	3	−3	−4	2

a Determine the play safe strategy for each player.
b Verify that there is no stable solution for this game.
c State the value of the game for Claire if both players play safe.
d State the value of the game for David if both players play safe.
e Determine the pay-off matrix for David.

Solution:

a

	D plays 1	D plays 2	D plays 3	D plays 4	Row min	
C plays 1	7	2	−3	5	−3	
C plays 2	4	−1	1	3	−1	←
C plays 3	−2	5	2	−1	−2	
C plays 4	3	−3	−4	2	−4	
Column max	7	5	2	5		
			↑			

C plays 2 (row maximin $= -1$)
D plays 3 (column minimax $= 2$)

b $-1 \neq 2$
row maximin ≠ column minimax
so no stable solution

c If C plays 2 and D plays 3, the value of the game is 1 to Claire

d *either* since the value of the game is 1 to Claire and it is a zero-sum game, the value of the game must be −1 to David
or
If C plays 2 and D plays 3 Claire wins 1, so David wins −1

e

	C plays 1	C plays 2	C plays 3	C plays 4
D plays 1	−7	−4	2	−3
D plays 2	−2	1	−5	3
D plays 3	3	−1	−2	4
D plays 4	5	−3	1	−2

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Exercise A, Question 5

Question:

Hilary and Denis play a two person zero-sum game, which is represented by the following pay-off matrix for Hilary.

	D plays 1	D plays 2	D plays 3	D plays 4	D plays 5
H plays 1	2	1	0	0	2
H plays 2	4	0	0	0	2
H plays 3	1	4	−1	−1	3
H plays 4	1	1	−1	−2	0
H plays 5	0	−2	−3	−3	−1

a Determine the play safe strategy for each player.
b Verify that there is a stable solution for this game and state the saddle points.
c State the value of the game for Hilary if both players play safe.
d State the value of the game for Denis if both players play safe.
e Determine the pay-off matrix for Denis.

Solution:

	D plays 1	D plays 2	D plays 3	D plays 4	D plays 5	Row min
H plays 1	2	1	0	0	2	0 ←
H plays 2	4	0	0	0	2	0 ←
H plays 3	1	4	−1	−1	3	−1
H plays 4	1	1	−1	−2	0	−2
H plays 5	0	−2	−3	−3	−1	−3
Column max	4	4	0	0	3	
			↑	↑		

a H plays 1 or 2
D plays 3 or 4

b row maximin = column minimax

$0 = 0$

so game stable

saddle points (H1, D3) (H2, D3) (H1, D4) (H2, D4)

c The value of the game to Hilary $= 0$

d The value of the game to Denis $= 0$

e

	H plays 1	H plays 2	H plays 3	H plays 4	H plays 5
D plays 1	−2	−4	−1	−1	0
D plays 2	−1	0	−4	−1	2
D plays 3	0	0	1	1	3
D plays 4	0	0	1	2	3
D plays 5	−2	−2	−3	0	1

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Exercise B, Question 1

Question:

	Freya plays 1	Freya plays 2
Ellie plays 1	1	−5
Ellie plays 2	−1	6
Ellie plays 3	3	−3

Ellie and Freya play a zero-sum game, represented by the pay-off matrix for Ellie shown above. Use dominance to reduce the game to a 2×2 game. You must make your reasoning clear.

Solution:

Row 3 dominates row 1 $(3 > 1, -3 > -5)$ so game can be reduced to

Ellie would always choose to play row 3 over row 1

	Freya plays 1	Freya plays 2
Ellie plays 2	−1	6
Ellie plays 3	3	−3

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Exercise B, Question 2

Question:

	Harry plays 1	Harry plays 2	Harry plays 3
Doug plays 1	−5	2	−1
Doug plays 2	2	−3	−6

Doug and Harry play a zero-sum game, represented by the pay-off matrix for Doug shown above. Use dominance to reduce the game to a 2×2 game. You must make your reasoning clear.

Solution:

Column 3 dominates 2 $(-1 < 2 \quad -6 < -3)$

	Harry plays 1	Harry plays 3
Doug plays 1	−5	−1
Doug plays 2	2	−6

Harry would always choose to play 3 over 1

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Exercise B, Question 3

Question:

	Nick plays 1	Nick plays 2	Nick plays 3
Chris plays 1	1	2	3
Chris plays 2	−1	−3	1
Chris plays 3	2	−1	5

Chris and Nick play a zero-sum game, represented by the pay-off matrix for Chris shown above. Use dominance to reduce the game to a 2×2 game. You must make your reasoning clear.

Solution:

Row 1 dominates row 2 $(1 > -1, 2 > -3, 3 > 1)$

Chris would always choose to play 1 over 2

Column 1 (or column 2) dominates column 3

$(1 < 3, -1 < 1, 2 < 5$ or $2 < 3, -3 < 1, -1 < 5$

	Nick plays 1	Nick plays 2
Chris plays 1	1	2
Chris plays 3	2	−1

Nick would always choose 1 (or 2) over 3

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Exercise B, Question 4

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to A.
c Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	2	−4
A plays 2	−1	3

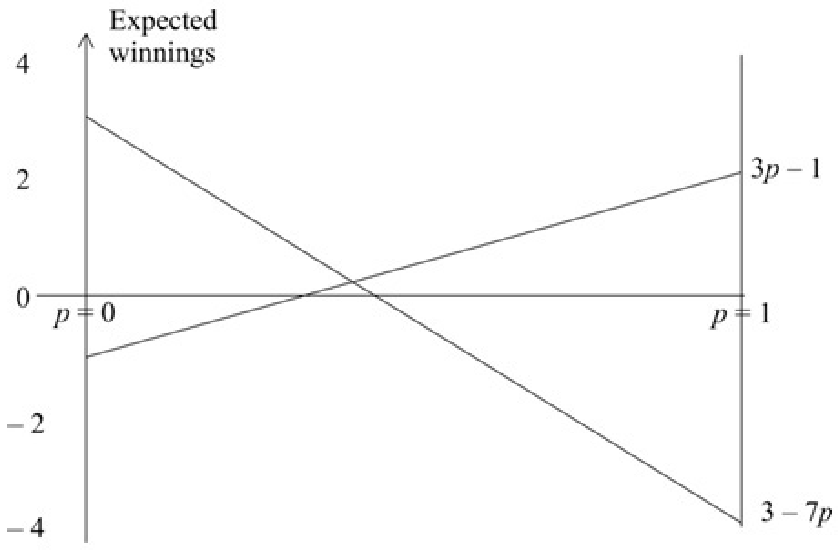
Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	2	−4	−4	
A plays 2	−1	3	−1	←
Column max	2	3		
	↑			

Since $2 \neq -1$ (column minimax $\neq$ row maximin) the game is not stable

b Let A play 1 with probability p
So A plays 2 with probability $(1-p)$
If B plays 1 A's expected winning are $2p-1(1-p)=3p-1$
If B plays 2 A's expected winnings are $4p+3(1-p)=3-7p$



$$3p-1=3-7p$$
$$10p=4$$
$$p=\frac{2}{5}$$

A should play 1 with probability $\frac{2}{5}$

A should play 2 with probability $\frac{3}{5}$

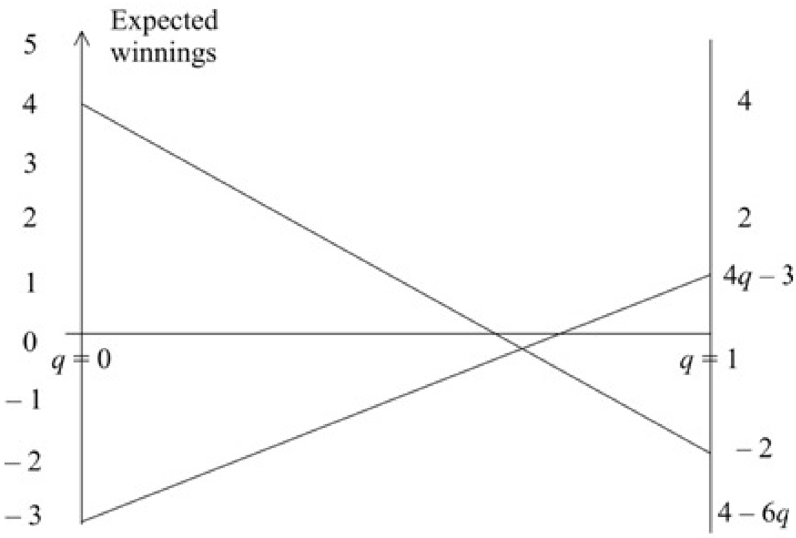
The value of the game to A is $3(\frac{2}{5})-1=\frac{1}{5}$

c Let B play 1 with probability q

so B plays 2 with probability $(1-q)$

If A plays 1 B's expected winnings are $-[2q-4(1-q)]=4-6q$

If A plays 2 B's expected winnings are $-[-q+3(1-q)]=4q-3$



$$4-6q=4q-3$$
$$10q=7$$
$$q=\frac{7}{10}$$

B should play 1 with probability $\frac{7}{10}$

B should play 2 with probability $\frac{3}{10}$

The value of the game to B is $4(\frac{3}{10})-3=\frac{-1}{5}$

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Exercise B, Question 5

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to A.
c Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	−3	5
A plays 2	2	−4

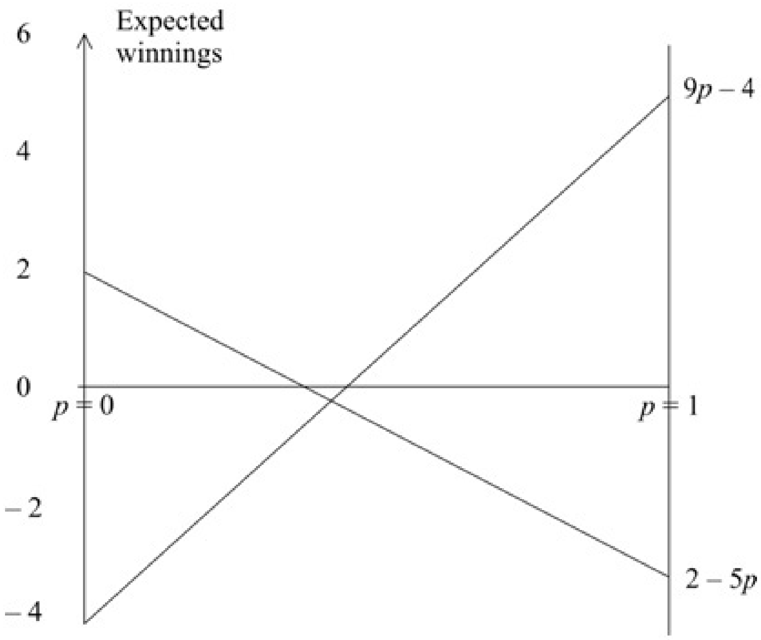
Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	−3	5	−3	←
B plays 2	2	−4	−4	
Column max	2	5		
	↑			

Since $2 \neq -3$ (column minimax $\neq$ row maximin) the game is not stable

b Let A play row 1 with probability p
So A plays row 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $-3p+2(1-p)=2-5p$
If B plays 2 A's expected winnings are $5p-4(1-p)=9p-4$



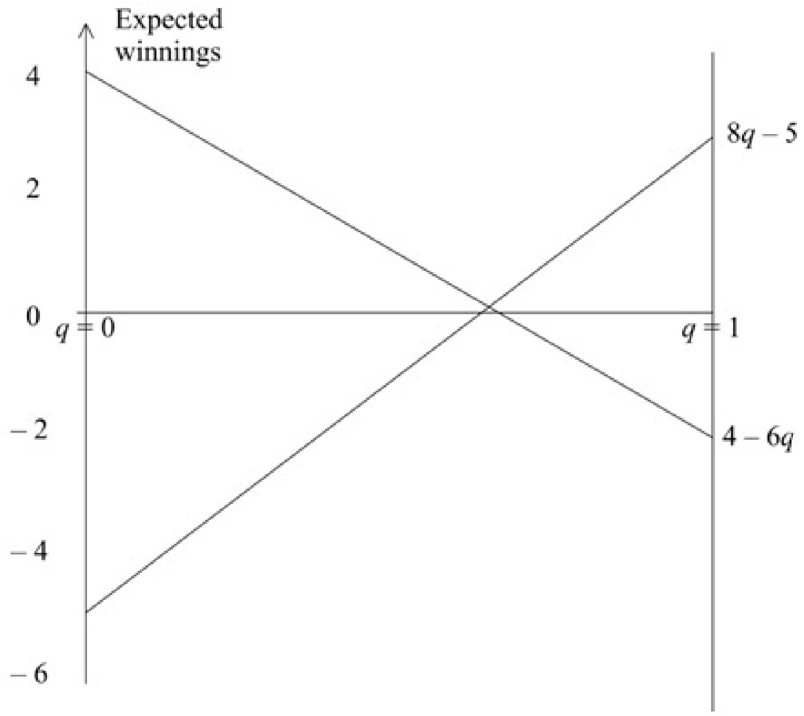
$$2-5p=9p-4$$
$$14p=6$$
$$p=\frac{3}{7}$$

A should play 1 with probability $\frac{3}{7}$

A should play 2 with proability $\frac{4}{7}$

The value of the game to A is $2-5\left(\frac{3}{7}\right)=\frac{-1}{7}$

c Let B play column 1 with probability q
So B plays column 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[-3q+5(1-q)]=8q-5$
If A plays 2 B's expected winning are $-[2q-4(1-q)]=4-6q$



$$8q - 5 = 4 - 6q$$
$$14q = 9$$
$$q = \frac{9}{14}$$

B should play 1 with probability $\frac{9}{14}$

B should play 2 with probability $\frac{5}{14}$

The value of the game to B is $8(\frac{9}{14}) - 5 = \frac{1}{7}$.

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Exercise B, Question 6

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to A.
c Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	5	−1
A plays 2	−2	1

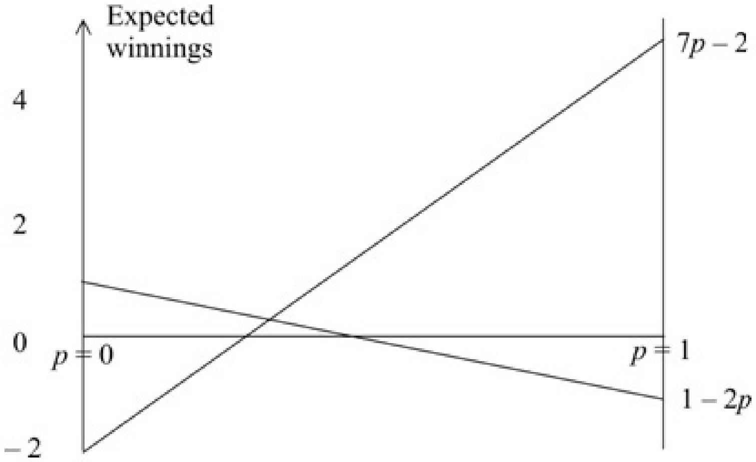
Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	5	−1	−1	←
A plays 2	−2	1	−2	
Column max	5	1		
		↑		

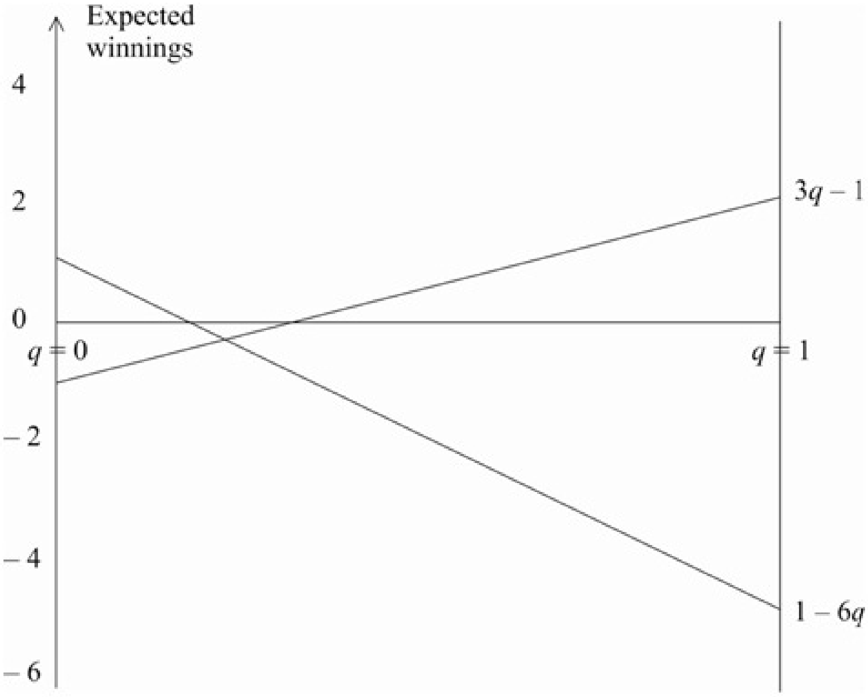
Since $-1 \neq 1$ (column minimax $\neq$ row maximin) the game is not stable

b Let A play row 1 with probability p
So A plays rows 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $5p-2(1-p)=7p-2$
If B plays 2 A's expected winnings are $-p+1(1-p)=1-2p$



$7p-2=1-2p$
$9p=3$
$p=\frac{1}{3}$
A should play 1 with probability $\frac{1}{3}$
A should play 2 with probability $\frac{2}{3}$
The value of the game to A is $7(\frac{1}{3})-2=\frac{1}{3}$

c Let B play column 1 with probability q
so B plays column 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[5q-1(1-q)]=1-6q$
If A plays 2 B's expected winning are $-[-2q+1(1-q)]=3q-1$



$$3q-1=1-6q$$
$$9q=2$$
$$q=\frac{2}{9}$$

B should play 1 with probability $\frac{2}{9}$

B should play 2 with probability $\frac{7}{9}$

The value of the game to B is

$$1-6\left(\frac{2}{9}\right)=-\frac{1}{3}$$

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Exercise B, Question 7

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to A.
c Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	−1	3
A plays 2	1	−2

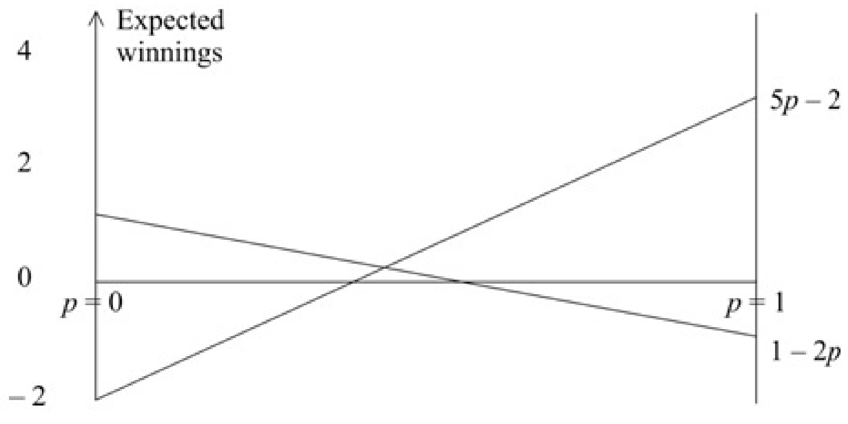
Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	−1	3	−1	←
A plays 2	1	−2	−2	
Column max	1	3		
	↑			

Since $1 \neq -1$ (column minimax $\neq$ row maximin) the game is not stable

b Let A play 1 with probability p
So A plays 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $-p+(1-p)=1-2p$
If B plays 2 A's expected winnings are $3p-2(1-p)=5p-2$



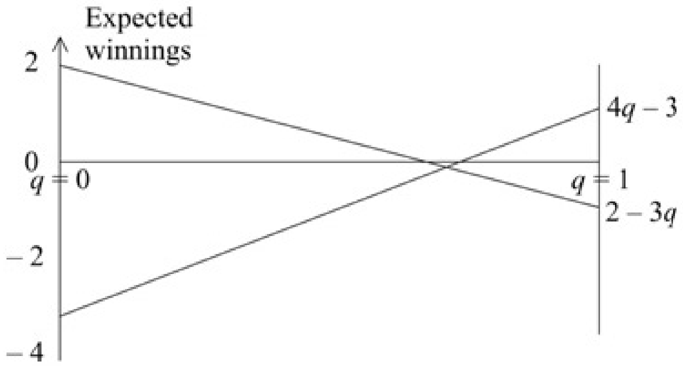
$$5p-2=1-2p$$
$$7p=3$$
$$p=\frac{3}{7}$$

A should play 1 with probability $\frac{3}{7}$

A should play 2 with probability $\frac{4}{7}$

The value of the game to A is $1-2(\frac{3}{7})=\frac{1}{7}$

c Let B play 1 with probability q
So B plays 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[-q+3(1-q)]=4q-3$
If A plays 2 B's expected winnings are $-[q-2(1-q)]=2-3q$



$$4q-3=2-3q$$
$$7q=5$$
$$q=\frac{5}{7}$$

B should play 1 with probability $\frac{5}{7}$

B should play 2 with probability $\frac{2}{7}$

The value of the game to B is $4\left(\frac{5}{7}\right)-3=-\frac{1}{7}$

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Exercise C, Question 1

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to A.

	B plays 1	B plays 2	B plays 3
A plays 1	−5	2	2
A plays 2	1	−3	−4

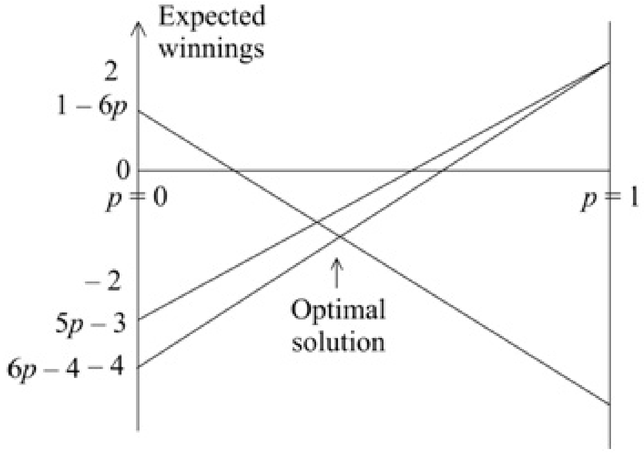
Solution:

a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	−5	2	2	−5	
A plays 2	1	−3	−4	−4	←
Column max	1	2	2		
	↑				

Since $1 \neq -4$ (column minimax $\neq$ row maximin) the game is not stable

b Let A play 1 with probability p
So A plays 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $-5p+1(1-p)=1-6p$
If B plays 2 A's expected winnings are $2p-3(1-p)=5p-3$
If B plays 3 A's expected winnings are $2p-4(1-p)=6p-4$



$6p-4=1-6p$
$12p=5$
$p=\frac{5}{12}$

A should play 1 with probability $\frac{5}{12}$
A should play 2 with probability $\frac{7}{12}$
The value of the game to A is
$1-6(\frac{5}{12})=-\frac{3}{2}$

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Exercise C, Question 2

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to A.

	B plays 1	B plays 2	B plays 3
A plays 1	2	6	−2
A plays 2	−1	−4	3

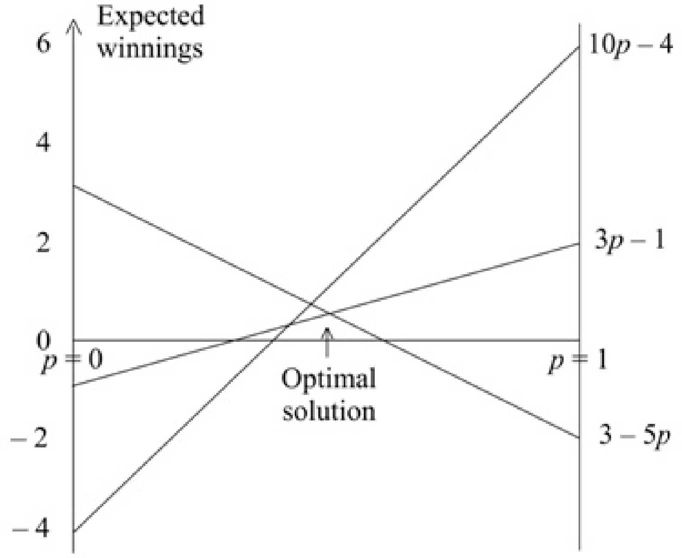
Solution:

a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	2	6	−2	−2	←
A plays 2	−1	−4	3	−4	
Column max	2	6	3		
	↑				

Since $2 \neq -2$ (column minimax $\neq$ row maximin) the game is not stable

b Let A play 1 with probability p
So A plays 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $2p-(1-p)=3p-1$
If B plays 2 A's expected winnings are $6p-4(1-p)=10p-4$
If B plays 3 A's expected winnings are $-2p+3(1-p)=3-5p$



$$3p-1=3-5p$$
$$8p=4$$
$$p=\frac{1}{2}$$

A should play 1 with probability $\frac{1}{2}$
A should play 2 with probability $\frac{1}{2}$
The value of the game to A is $3(\frac{1}{2})-1=\frac{1}{2}$

Solutionbank D2

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Exercise C, Question 3

Question:

a Verify that there is no stable solution.

b Determine the optimal mixed strategy and the value of the game to A.

	B plays 1	B plays 2	B plays 3
A plays 1	-2	3	6
A plays 2	5	1	-4

Solution:

a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	-2	3	6	-2	$\leftarrow$
A plays 2	5	1	-4	-4	
Column max	5	3	6		
		$\uparrow$			

Since $3 \neq -2$ (column minimax $\neq$ row maximin) the game is not stable

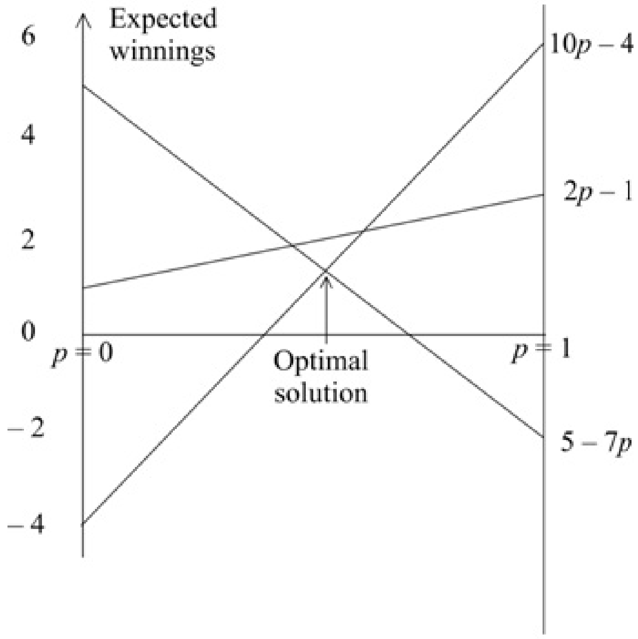
b Let A play 1 with probability p

So A plays 2 with probability $(1-p)$

If B plays 1 A's expected winnings are $-2p+5(1-p)=5-7p$

If B plays 2 A's expected winnings are $3p+1(1-p)=2p+1$

If B plays 3 A's expected winnings are $6p-4(1-p)=10p-4$



$$10p-4=5-7p$$
$$17p=9$$
$$p=\frac{9}{17}$$

A should play 1 with probability $\frac{9}{17}$

A should play 2 with probability $\frac{8}{17}$

The value of the game to A is

$10(\frac{9}{17})-4=\frac{22}{17}$

Solutionbank D2
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Exercise C, Question 4

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to A.

	B plays 1	B plays 2	B plays 3
A plays 1	5	−2	−4
A plays 2	−3	1	6

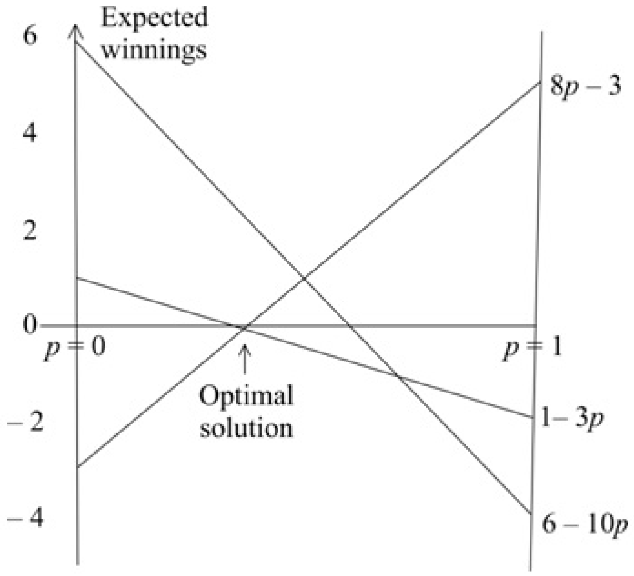
Solution:

a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	5	−2	−4	−4	
A plays 2	−3	1	6	−3	←
Column max	5	1	6		
		↑			

Since $1 \neq -3$ (column minimax ≠ row maximin) the game is not stable

b Let A play 1 with probability p
So A plays 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $5p-3(1-p)=8p-3$
If B plays 2 A's expected winnings are $-2p+1(1-p)=1-3p$
If B plays 3 A's expected winnings are $-4p+6(1-p)=6-10p$



$8p-3=1-3p$
$11p=4$
$p=\frac{4}{11}$

A should play 1 with probability $\frac{4}{11}$
A should play 2 with probability $\frac{7}{11}$
The value of the game to A is
$1-3\left(\frac{4}{11}\right)=-\frac{1}{11}$

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Exercise C, Question 5

Question:

a Verify that there is no stable solution.

b Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	−1	1
A plays 2	3	−4
A plays 3	−2	2

Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	−1	1	−1	←
A plays 2	3	−4	−4	
A plays 3	−2	2	−2	
Column max	3	2		
		↑		

Since $2 \neq -1$ (column minimax $\neq$ row maximin) the game is not stable

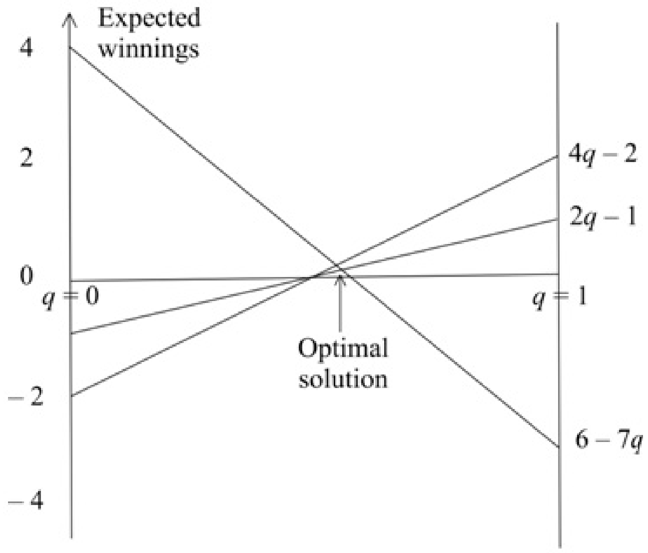
b Let B play 1 with probability q

So B plays 2 with probability $(1-q)$

If A plays 1 B's expected winnings are $-[-q+1(1-q)] = 2q-1$

If A plays 2 B's expected winnings are $-[3q-4(1-q)] = 4-7q$

If A plays 3 B's expected winnings are $-[-2q+2(1-q)] = 4q-2$



$$2q-1 = 4-7q$$
$$9q = 5$$
$$q = \frac{5}{9}$$

B should play 1 with probability $\frac{5}{9}$

B should play 2 with probability $\frac{4}{9}$

The value of the game to B is $2(\frac{5}{9})-1=\frac{1}{9}$

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 6

Question:

a Verify that there is no stable solution.

b Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	−5	4
A plays 2	3	−3
A plays 3	1	−2

Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	−5	4	−5	
A plays 2	3	−3	−3	
A plays 3	1	−2	−2	←
Column max	3	4		
	↑			

Since $3 \neq -2$ (column minimax $\neq$ row maximin) the game is not stable

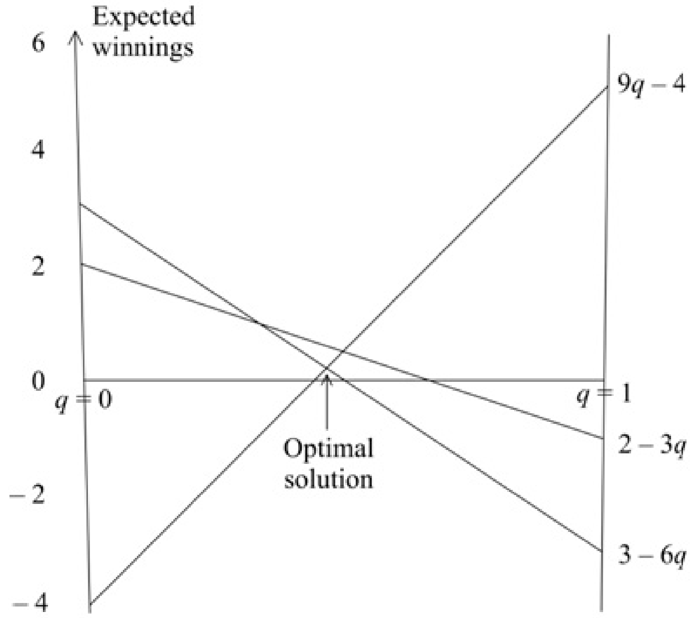
b Let B play 1 with probability q

So B plays 2 with probability $(1-q)$

If A plays 1 B's expected winnings are $-[-5q+4(1-q)] = 9q-4$

If A plays 2 B's expected winnings are $-[3q-3(1-q)] = 3-6q$

If A plays 3 B's expected winnings are $-[q-2(1-q)] = 2-3q$



$$9q-4 = 3-6q$$
$$15q = 7$$
$$q = \frac{7}{15}$$

B should play 1 with probability $\frac{7}{15}$

B should play 2 with probability $\frac{8}{15}$

The value of the game to B is $9(\frac{7}{15})-4 = \frac{3}{15}$

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Edexcel AS and A Level Modular Mathematics

Exercise C, Question 7

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	-3	2
A plays 2	-1	-2
A plays 3	2	-4

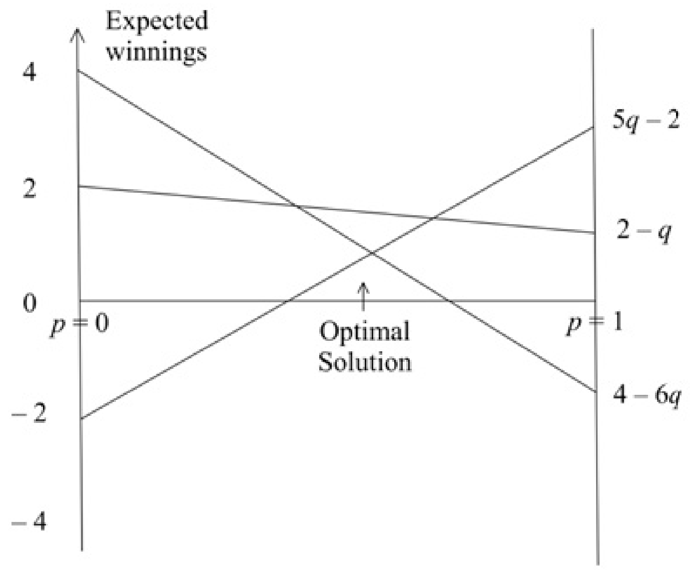
Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	-3	2	-3	
A plays 2	-1	-2	-2	←
A plays 3	2	-4	-4	
Column max	2	2		
	↑	↑		

Since $2 \neq -2$ (column minimax $\neq$ row maximin) the game is not stable

b Let B play 1 with probability q
So B plays 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[-3q+2(1-q)] = 5q-2$
If A plays 2 B's expected winnings are $-[-q-2(1-q)] = 2-q$
If A plays 3 B's expected winnings are $-[2q-4(1-q)] = 4-6q$



$$5q-2 = 4-6q$$
$$11q = 6$$
$$q = \frac{6}{11}$$

B should play 1 with probability $\frac{6}{11}$

B should play 2 with probability $\frac{5}{11}$

The value of the game to B is $5(\frac{6}{11})-2 = \frac{8}{11}$

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 8

Question:

a Verify that there is no stable solution.
b Determine the optimal mixed strategy and the value of the game to B.

	B plays 1	B plays 2
A plays 1	2	−3
A plays 2	−2	4
A plays 3	1	−1

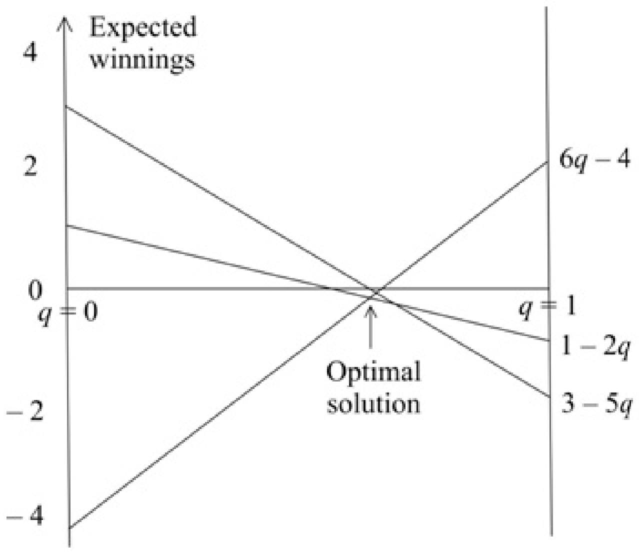
Solution:

a

	B plays 1	B plays 2	Row min	
A plays 1	2	−3	−3	
A plays 2	−2	4	−2	
A plays 3	1	−1	−1	←
Column max	2	4		
	↑			

Since $2 \neq -1$ (column minimax $\neq$ row maximin) the game is not stable

b Let B play 1 with probability q
So B plays 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[2q-3(1-q)]=3-5q$
If A plays 2 B's expected winnings are $-[-2q+4(1-q)]=6q-4$
If A plays 3 B's expected winnings are $-[q-1(1-q)]=1-2q$



$6q-4=1-2q$
$8q=5$
$q=\frac{5}{8}$

B should play 1 with probability $\frac{5}{8}$

B should play 2 with probability $\frac{3}{8}$

The value of the game to B is $6(\frac{5}{8})-4=-\frac{1}{4}$

Solutionbank D2

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Exercise D, Question 1

Question:

Formulate the game below as a linear programming problems for player A, writing the constraints as equalities and clearly defining your variables.

$$\begin{array}{r|cc|} & \text{B plays 1} & \text{B plays 2} \\ \text{A plays 1} & -1 & 1 \\ \text{A plays 2} & 3 & -4 \\ \text{A plays 3} & -2 & 2 \end{array}$$

Solution:

Add 5 to all elements

	B plays 1	B plays 2
A plays 1	4	6
A plays 2	8	1
A plays 3	3	7

Let A play 1 with probability p_1
and A play 2 with probability p_2
and A play 3 with probability p_3
Let the value of the game to A be v and $V = v + 5$

Maximise $P = V$

Subject to
$$4p_1 + 8p_2 + 3p_3 \geq V \Rightarrow V - 4p_1 - 8p_2 - 3p_3 + r = 0$$
$$6p_1 + p_2 + 7p_3 \geq V \Rightarrow V - 6p_2 - p_2 - 7p_3 + s = 0$$
$$p_1 + p_2 + p_3 \leq 1 \Rightarrow p_1 + p_2 + p_3 + t = 1$$
$$p_1, p_2, p_3, r, s, t \geq 0$$

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Exercise D, Question 2

Question:

Formulate the game below as a linear programming problems for player A, writing the constraints as equalities and clearly defining your variables.

$$\begin{array}{c|ccc|} & \text{B plays 1} & \text{B plays 2} & \text{B plays 3} \\ \text{A plays 1} & -5 & 4 & 1 \\ \text{A plays 2} & 3 & -3 & 2 \\ \text{A plays 3} & 1 & -2 & -1 \end{array}$$

Solution:

Add 6 to all elements

	B plays 1	B plays 2	B plays 3
A plays 1	1	10	7
A plays 2	9	3	8
A plays 3	7	4	5

Let A play 1 with probability p_1
and A play 2 with probability p_2
and A play 3 with probability p_3
Let the value of the game to A be v and $V = v + 6$

Maximise $P = V$

Subject to

$$\begin{aligned} p_1 + 9p_2 + 7p_3 \geq V \Rightarrow V - p_1 - 9p_2 - 7p_2 + r &= 0 \\ 10p_1 + 3p_2 + 4p_3 \geq V \Rightarrow V - 10p_1 - 3p_2 - 4p_3 + s &= 0 \\ 7p_1 + 8p_2 + 5p_3 \geq V \Rightarrow V - 7p_1 - 8p_2 - 5p_3 + t &= 0 \\ p_1 + p_2 + p_3 \leq 1 \Rightarrow p_1 + p_2 + p_3 + u &= 1 \\ p_1, p_2, p_3, r, s, t, u &\geq 0 \end{aligned}$$

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Exercise D, Question 3

Question:

Formulate the game below as a linear programming problems for player A, writing the constraints as equalities and clearly defining your variables.

$$\begin{array}{c|ccc|} & \text{B plays 1} & \text{B plays 2} & \text{B plays 3} \\ \text{A plays 1} & -3 & 2 & -1 \\ \text{A plays 2} & -1 & -2 & 1 \\ \text{A plays 3} & 2 & -4 & -2 \end{array}$$

Solution:

Add 5 to all elements

	B plays 1	B plays 2	B plays 3
A plays 1	2	7	4
A plays 2	4	3	6
A plays 3	7	1	3

Let A play 1 with probability p_1

Let A play 2 with probability p_2

Let A play 3 with probability p_3

Let the value of the game to A be v and $V = v + 5$

Maximise $P = V$

Subject to

$$\begin{aligned} 2p_1 + 4p_2 + 7p_3 \geq V \Rightarrow V - 2p_1 - 4p_2 - 7p_3 + r &= 0 \\ 7p_1 + 3p_2 + p_3 \geq V \Rightarrow V - 7p_1 - 3p_2 - p_3 + s &= 0 \\ 4p_1 + 6p_2 + 3p_3 \geq V \Rightarrow V - 4p_1 - 6p_2 - 3p_3 + t &= 0 \\ p_1 + p_2 + p_3 \leq 1 \Rightarrow p_1 + p_2 + p_3 + u &= 1 \\ p_1, p_2, p_3, r, s, t, u &\geq 0 \end{aligned}$$

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Exercise D, Question 4

Question:

Formulate the game below as a linear programming problems for player A, writing the constraints as equalities and clearly defining your variables.

	B plays 1	B plays 2	B plays 3
A plays 1	2	−3	−1
A plays 2	−2	4	1
A plays 3	1	−1	0

Solution:

Add 4 to all elements

	B plays 1	B plays 2	B plays 3
A plays 1	6	1	3
A plays 2	2	8	5
A plays 3	5	3	4

Let A play 1 with probability p_1

Let A play 2 with probability p_2

Let A play 3 with probability p_3

Let the value of the game to A be v and $V = v + 4$

Maximise $P = V$

Subject to

$$6p_1 + 2p_2 + 5p_3 \geq V \Rightarrow V - 6p_1 - 2p_2 - 5p_3 + r = 0$$
$$p_1 + 8p_2 + 3p_3 \geq V \Rightarrow V - p_1 - 8p_2 - 3p_3 + s = 0$$
$$3p_1 + 5p_2 + 4p_3 \geq V \Rightarrow V - 3p_1 - 5p_2 - 4p_3 + t = 0$$
$$p_1 + p_2 + p_3 \leq 1 \Rightarrow p_1 + p_2 + p_3 + u = 1$$
$$p_1, p_2, p_3, r, s, t, u \geq 0$$

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Exercise D, Question 5

Question:

Formulate the game below as a linear programming problem for player B, writing the constraints as equalities and clearly defining your variables.

	B plays 1	B plays 2	B plays 3
A plays 1	−5	2	3
A plays 2	1	−3	−4

Solution:

	A plays 1	A plays 2			A plays 1	A plays 2
B plays 1	5	−1	Adding	B plays 1	9	3
B plays 2	−2	3	4 to all	B plays 2	2	7
B plays 3	−3	4	elements	B plays 3	1	8

Let B play 1 with probability q_1

Let B play 2 with probability q_2

Let B play 3 with probability q_3

Let the value of the game to B be v and $V = v + 4$

Maximise $P = V$

Subject to

$$9q_1 + 2q_2 + q_3 \geq V \quad V - 9q_1 - 2q_2 - q_3 + r = 0$$

$$3q_1 + 7q_2 + 8q_3 \geq V \quad V - 3q_1 - 7q_2 - 8q_3 + s = 0$$

$$q_1 + q_2 + q_3 \leq 1 \quad q_1 + q_2 + q_3 + t = 1$$

$$q_1, q_2, q_3, r, s, t \geq 0$$

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Exercise D, Question 6

Question:

Formulate the game below as a linear programming problems for player B, writing the constraints as equalities and clearly defining your variables.

	B plays 1	B plays 2	B plays 3
A plays 1	−5	4	1
A plays 2	3	−3	2
A plays 3	1	−2	−1

Solution:

	A plays 1	A plays 2	A plays 3			A plays 1	A plays 2	A plays 3
B plays 1	5	−3	−1	Adding 5	B plays 1	10	2	4
B plays 2	−4	3	2	to all	B plays 2	1	8	7
B plays 3	−1	−2	1	elements	B plays 3	4	3	6

Let B play 1 with probability q_1

Let B play 2 with probability q_2

Let B play 3 with probability q_3

Let the value of the game to B be v and $V = v + 5$

Maximise $P = V$

Subject to

$$10q_1 + q_2 + 4q_3 \geq V \Rightarrow V - 10q_1 - q_2 - 4q_3 + r = 0$$
$$2q_1 + 8q_2 + 3q_3 \geq V \Rightarrow V - 2q_1 - 8q_2 - 3q_3 + s = 0$$
$$4q_1 + 7q_2 + 6q_3 \geq V \Rightarrow V - 4q_1 - 7q_2 - 6q_3 + t = 0$$
$$q_1 + q_2 + q_3 \leq 1 \Rightarrow q_1 + q_2 + q_3 + u = 1$$
$$q_1, q_2, q_3, r, s, t, u \geq 0$$

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Exercise D, Question 7

Question:

Formulate the game below as a linear programming problems for player B, writing the constraints as equalities and clearly defining your variables.

	B plays 1	B plays 2	B plays 3
A plays 1	−3	2	−1
A plays 2	−1	−2	1
A plays 3	2	−4	−2

Solution:

	A plays 1	A plays 2	A plays 3			A plays 1	A plays 2	A plays 3
B plays 1	3	1	−2	Adding 3	B plays 1	6	4	1
B plays 2	−2	2	4	to all	B plays 2	1	5	7
B plays 3	1	−1	2	elements	B plays 3	4	2	5

Let B play 1 with probability q_1

Let B play 2 with probability q_2

Let B play 3 with probability q_3

Let the value of the game to B be v and $V = v+3$

Maximise $P = V$

Subject to:

$$6q_1 + q_2 + 4q_3 \geq V \Rightarrow V - 6q_1 - q_2 - 4q_3 + r = 0$$
$$4q_1 + 5q_2 + 2q_3 \geq V \Rightarrow V - 4q_1 - 5q_2 - 2q_3 + s = 0$$
$$q_1 + 7q_2 + 5q_3 \geq V \Rightarrow V - q_1 - 7q_2 - 5q_3 + t = 0$$
$$q_1 + q_2 + q_3 \leq 1 \Rightarrow q_1 + q_2 + q_3 + u = 1$$
$$q_1, q_2, q_3, r, s, t, u \geq 0$$

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Exercise D, Question 8

Question:

Formulate the game below as a linear programming problems for player B, writing the constraints as equalities and clearly defining your variables.

	B plays 1	B plays 2	B plays 3
A plays 1	2	−3	−1
A plays 2	−2	4	1
A plays 3	1	−1	0

Solution:

	A plays 1	A plays 2	A plays 3			A plays 1	A plays 2	A plays 3
B plays 1	−2	2	−1	Adding	B plays 1	3	7	4
B plays 2	3	−4	1	5 to all	B plays 2	8	1	6
B plays 3	1	−1	0	elements	B plays 3	6	4	5

Let B play 1 with probability q_1

Let B play 2 with probability q_2

Let B play 3 with probability q_3

Let the value of the game to B be v and $V = v + 5$

Maximise $P = V$

Subject to:

$$3q_1 + 8q_2 + 6q_3 \geq v \Rightarrow V - 3q_1 - 8q_2 - 6q_3 + r = 0$$
$$7q_1 + q_2 + 4q_3 \geq V \Rightarrow V - 7q_1 - q_2 - 4q_3 + s = 0$$
$$4q_1 + 6q_2 + 5q_3 \geq 1v \Rightarrow V - 4q_1 - 6q_2 - 5q_3 + t = 0$$
$$q_1 + q_2 + q_3 \leq 1 \Rightarrow q_1 + q_2 + q_3 + u = 1$$
$$q_1, q_2, q_3, r, s, t, u \geq 0$$

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Exercise D, Question 9

Question:

Using your answer to question **1**,

a write down an initial simplex tableau to solve the zero-sum game below, for player A,

b use the simplex algorithm to determine A's best strategy.

	B plays 1	B plays 2
A plays 1	−1	1
A plays 2	3	−4
A plays 3	−2	2

Solution:

a

b.v.	V	p_1	p_2	p_3	r	s	t	value
r	1	−4	−8	−3	1	0	0	0
s	1	−6	−1	−7	0	1	0	0
t	0	1	1	1	0	0	1	1
P	−1	0	0	0	0	0	0	0

b

b.v.	V	p_1	p_2	p_3	r	s	t	value	
V	1	−4	−8	−3	1	0	0	0	R1 ÷ 1
s	0	−2	7	−4	−1	1	0	0	R2 − R1
t	0	1	1	1	0	0	1	1	R3 no change
P	0	−4	−8	−3	1	0	0	0	R4 + R1

b.v.	V	p_1	p_2	p_3	r	s	t	values	
V	1	0	−4	1	1	0	3	3	R1+4R3
s	0	0	9	−2	−1	1	2	2	R2+2R3
p_1	0	1	1	1	0	0	1	1	R3 no change
P	0	0	−4	1	1	0	4	4	R4 +4R3

b.v.	V	p_1	p_2	p_3	r	s	t	value	
V	1	0	0	$\frac{1}{9}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{35}{9}$	$\frac{44}{9}$	R1+4R2
p_2	0	0	1	$\frac{-2}{9}$	$\frac{-1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	R2 ÷ 9
p_1	0	1	0	$\frac{11}{9}$	$\frac{1}{9}$	$\frac{-1}{9}$	$\frac{7}{9}$	$\frac{7}{9}$	R3 − R2
P	0	0	0	$\frac{1}{9}$	$\frac{5}{9}$	$\frac{4}{9}$	$\frac{35}{9}$	$\frac{44}{9}$	R4 +4R2

$V = \frac{44}{9}$ so $v = \frac{44}{9} - 5 = \frac{-1}{9}$ $\quad p_1 = \frac{7}{9} \quad p_2 = \frac{2}{9} \quad p_3 = 0$

A should play 1 with probability $\frac{7}{9}$, play 2 with probability $\frac{2}{9}$ and play 3 never

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Exercise D, Question 10

Question:

Using your answer to question **5**,

a write down an initial simplex tableau to solve the zero-sum game below, for player B,

b use the simplex algorithm to determine B's best strategy.

	B plays 1	B plays 2	B plays 3
A plays 1	-5	2	3
A plays 2	1	-3	-4

Solution:

a

b.v.	V	q_1	q_2	q_3	r	s	t	value
r	1	−9	−2	−1	1	0	0	0
s	1	−3	−7	−8	0	1	0	0
t	0	1	1	1	0	0	1	1
P	−1	0	0	0	0	0	0	0

b

b.v.	V	q_1	q_2	q_3	r	s	t	value	Row operations
V	1	−9	−2	−1	1	0	0	0	R2 ÷ 1
s	0	6	−5	−7	−1	1	0	0	R2 − R1
t	0	1	1	1	0	0	1	1	R3 no change
P	0	−9	−2	−1	1	0	0	0	R4 + R1

b.v.	V	q_1	q_2	q_3	r	s	t	value	Row operations
v	1	0	$\frac{-19}{2}$	$\frac{-23}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$	0	0	R1 + 9R2
q_1	0	1	$\frac{-5}{6}$	$\frac{-7}{6}$	$\frac{-1}{6}$	$\frac{1}{6}$	0	0	R2 ÷ 6
t	0	0	$\frac{11}{6}$	$\frac{13}{6}$	$\frac{1}{6}$	$\frac{-1}{6}$	1	1	R3 − R2
P	0	0	$\frac{-19}{2}$	$\frac{-23}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$	0	0	R4 + 9R2

b.v.	V	q_1	q_2	q_3	r	s	t	value	Row operations
v	1	0	$\frac{3}{13}$	0	$\frac{5}{13}$	$\frac{8}{13}$	$\frac{69}{13}$	$\frac{69}{13}$	R1 + $\frac{23}{2}$R3
q_1	0	1	$\frac{2}{13}$	0	$\frac{-1}{13}$	$\frac{1}{13}$	$\frac{7}{13}$	$\frac{7}{13}$	R2 + $\frac{7}{6}$R3
q_3	0	0	$\frac{11}{13}$	1	$\frac{1}{13}$	$\frac{-1}{13}$	$\frac{6}{13}$	$\frac{6}{13}$	R3 ÷ $\frac{15}{6}$
P	0	0	$\frac{3}{13}$	0	$\frac{5}{13}$	$\frac{8}{13}$	$\frac{69}{13}$	$\frac{69}{13}$	R4 + $\frac{23}{2}$R3

$V = \frac{69}{13}$ so $V = \frac{69}{13} - 4 = \frac{17}{13}$ $\quad q_1 = \frac{7}{13} \quad q_2 = 0 \quad q_3 = \frac{6}{13}$

B should play 1 with probability $\frac{7}{13}$, play 2 never and play 3 with probability $\frac{6}{13}$

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Exercise D, Question 11

Question:

	B plays 1	B plays 2	B plays 3
A plays 1	−5	4	1
A plays 2	3	−3	2
A plays 3	1	−2	−1

Using your answer to question **2**,

a write down an initial simplex tableau to solve the zero-sum game, for player A,

b use the simplex algorithm to determine A's best strategy.

Using your answer to question **6**,

c write down an initial simplex tableau to solve the zero-sum game, for player B,

d use the simplex algorithm to determine B's best strategy.

Solution:

a

b.v.	V	p_1	p_2	p_3	r	s	t	u	value
r	1	−1	−9	−7	1	0	0	0	0
s	1	−10	−3	−4	0	1	0	0	0
t	1	−7	−8	−5	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	−1	0	0	0	0	0	0	0	0

b

b.v.	V	p_1	p_2	p_3	r	s	t	u	value	Row operations
V	1	−1	−9	−7	1	0	0	0	0	R1÷1
s	0	−9	6	3	−1	1	0	0	0	R2−R1
t	0	−6	1	2	−1	0	1	0	0	R3−R1
u	0	1	1	1	0	0	0	1	1	R4 no change
P	0	−1	−9	−7	1	0	0	0	0	R5+R1

b.v.	V	p_1	p_2	p_3	r	s	t	u	value	Row operations
V	1	$\frac{-29}{2}$	0	$\frac{-5}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$	0	0	0	R1+9R2
p_2	0	$\frac{-3}{2}$	1	$\frac{1}{2}$	$\frac{-1}{6}$	$\frac{1}{6}$	0	0	0	R2÷6
t	0	$\frac{-9}{2}$	0	$\frac{3}{2}$	$\frac{-5}{6}$	$\frac{-1}{6}$	1	0	0	R3−R2
u	0	$\frac{5}{2}$	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{-1}{6}$	0	1	1	R4−R2
P	0	$\frac{-29}{2}$	0	$\frac{-5}{2}$	$\frac{-1}{2}$	$\frac{3}{2}$	0	0	0	R5+9R2

b.v.	V	p_1	p_2	p_3	r	s	t	u	value	Row operations
V	1	0	0	$\frac{2}{5}$	$\frac{7}{15}$	$\frac{8}{15}$	0	$\frac{29}{5}$	$\frac{29}{5}$	$R1+\frac{29}{2}R4$
p_2	0	0	1	$\frac{4}{5}$	$\frac{-1}{15}$	$\frac{1}{15}$	0	$\frac{3}{5}$	$\frac{3}{5}$	$R2+\frac{3}{2}R4$
t	0	0	0	$\frac{12}{5}$	$\frac{-8}{15}$	$\frac{-7}{15}$	1	$\frac{9}{5}$	$\frac{9}{5}$	$R3+\frac{9}{2}R4$
p_1	0	1	0	$\frac{1}{5}$	$\frac{1}{15}$	$\frac{-1}{15}$	0	$\frac{2}{5}$	$\frac{2}{5}$	$R4\div\frac{5}{2}$
P	0	0	0	$\frac{2}{5}$	$\frac{7}{15}$	$\frac{8}{15}$	0	$\frac{29}{5}$	$\frac{29}{5}$	$R5+\frac{29}{2}R4$

$V=\frac{29}{5}$, so $v=\frac{29}{5}-6=\frac{-1}{5}$, $p_1=\frac{2}{5}$ $p_2=\frac{3}{5}$ $p_3=0$

A should play 1 with probability $\frac{2}{5}$

A should play 2 with probability $\frac{3}{5}$

A should play 3 never

c

b.v.	V	q_1	q_2	q_3	r	s	t	u	value
r	1	−10	−1	−4	1	0	0	0	0
s	1	−2	−8	−3	0	1	0	0	0
t	1	−4	−7	−6	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	−1	0	0	0	0	0	0	0	0

d

b.v.	V	q_1	q_2	q_3	r	s	t	u	value	Row operations
V	1	−10	−1	−4	1	0	0	0	0	R1 ÷ 1
s	0	8	−7	1	−1	1	0	0	0	R2 − R1
t	0	6	−6	−2	−1	0	1	0	0	R3 − R1
u	0	1	1	1	0	0	0	1	1	R4 no change
P	0	−10	−1	−4	1	0	0	0	0	R5 + R1

b.v.	V	q_1	q_2	q_3	r	s	t	u	value	Row operations
V	1	0	$\frac{-39}{4}$	$\frac{-11}{4}$	$\frac{-1}{4}$	$\frac{5}{4}$	0	0	0	R1 + 10R2
q_1	0	1	$\frac{-7}{8}$	$\frac{1}{8}$	$\frac{-1}{8}$	$\frac{1}{8}$	0	0	0	R2 ÷ 8
t	0	0	$\frac{-3}{4}$	$\frac{-11}{4}$	$\frac{-1}{4}$	$\frac{-3}{4}$	1	0	0	R3 − 6R2
u	0	0	$\frac{15}{8}$	$\frac{7}{8}$	$\frac{1}{8}$	$\frac{-1}{8}$	0	1	1	R4 − R2
P	0	0	$\frac{-39}{4}$	$\frac{-11}{4}$	$\frac{-1}{4}$	$\frac{5}{4}$	0	0	0	R5 + 10R2

b.v.	V	q_1	q_2	q_3	r	s	t	u	value	Row operations
V	1	0	0	$\frac{9}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	0	$\frac{26}{5}$	$\frac{26}{5}$	$\text{R1}+\frac{39}{4}\text{R4}$
q_1	0	1	0	$\frac{8}{15}$	$\frac{-1}{15}$	$\frac{1}{15}$	0	$\frac{7}{15}$	$\frac{7}{15}$	$\text{R2}+\frac{7}{8}\text{R4}$
t	0	0	0	$\frac{-6}{5}$	$\frac{-1}{5}$	$\frac{-4}{5}$	1	$\frac{2}{5}$	$\frac{6}{15}$	$\text{R3}+\frac{3}{4}\text{R4}$
q_2	0	0	1	$\frac{7}{15}$	$\frac{1}{15}$	$\frac{-1}{15}$	0	$\frac{8}{15}$	$\frac{8}{15}$	$\text{R4}\div\frac{15}{8}$
P	0	0	0	$\frac{9}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	0	$\frac{26}{5}$	$\frac{26}{5}$	$\text{R5}+\frac{39}{4}\text{R4}$

$V=\frac{26}{5}$, so $v=\frac{26}{5}-5=\frac{1}{5}$ $\quad q_1=\frac{7}{15}$ $\quad q_2=\frac{8}{15}$ $\quad q_3=0$

B should play 1 with probability $\frac{7}{15}$

B should play 2 with probability $\frac{8}{15}$

B should play 3 never

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Exercise D, Question 12

Question:

	B plays 1	B plays 2	B plays 3
A plays 1	−3	2	−1
A plays 2	−1	−2	1
A plays 3	2	−4	−2

Using your answer to question **3**,

a write down an initial simplex tableau to solve the zero-sum game, for player A,

b use the simplex algorithm to determine A's best strategy.

Using your answer to question 7,

c write down an initial simplex tableau to solve the zero-sum game, for player B,

d use the simplex algorithm to determine B's best strategy.

Solution:

a

b.v.	V	p_1	p_2	p_3	r	s	t	u	value
r	1	−2	−4	−7	1	0	0	0	0
s	1	−7	−3	−1	0	1	0	0	0
t	1	−4	−6	−3	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	−1	0	0	0	0	0	0	0	0

b

b.v.	V	p_1	p_2	p_3	r	s	t	u	value	Row operations
V	1	−2	−4	−7	1	0	0	0	0	R1 ÷ 1
s	0	−5	1	6	−1	1	0	0	0	R2 − R1
t	0	−2	−2	4	−1	0	1	0	0	R3 − R1
u	0	1	1	1	0	0	0	1	1	R4 no change
P	0	−2	−4	−7	1	0	0	0	0	R5 + R1

b.v.	V	p_1	p_2	p_3	r	s	t	u	value	Row operations
V	1	$\frac{-47}{6}$	$\frac{-17}{6}$	0	$\frac{-1}{6}$	$\frac{7}{6}$	0	0	0	R1 + 7R2
p_3	0	$\frac{-5}{6}$	$\frac{1}{6}$	1	$\frac{-1}{6}$	$\frac{1}{6}$	0	0	0	R2 ÷ 6
t	0	$\frac{4}{3}$	$\frac{-8}{3}$	0	$\frac{-1}{3}$	$\frac{-2}{3}$	1	0	0	R3 − 4R2
u	0	$\frac{11}{6}$	$\frac{5}{6}$	0	$\frac{1}{6}$	$\frac{-1}{6}$	0	1	1	R4 − R2
P	0	$\frac{-47}{6}$	$\frac{-17}{6}$	0	$\frac{-1}{6}$	$\frac{7}{6}$	0	0	0	R5 + 7R2

b.v.	V	p_1	p_2	p_3	r	s	t	u	value	Row operations
V	1	0	$\frac{-37}{2}$	0	$\frac{-17}{8}$	$\frac{-11}{4}$	$\frac{47}{8}$	0	0	$R1 \div \frac{47}{6} R3$
p_3	0	0	$\frac{-3}{2}$	1	$\frac{-3}{8}$	$\frac{-1}{4}$	$\frac{5}{8}$	0	0	$R2 + \frac{5}{6} R3$
p_1	0	1	−2	0	$\frac{-1}{4}$	$\frac{-1}{2}$	$\frac{3}{4}$	0	0	$R3 \div \frac{4}{3}$
u	0	0	$\frac{9}{2}$	0	$\frac{5}{8}$	$\frac{3}{4}$	$\frac{-11}{8}$	1	1	$R4 - \frac{11}{6} R3$
P	0	0	$\frac{-37}{2}$	0	$\frac{-17}{8}$	$\frac{-11}{4}$	$\frac{47}{8}$	0	0	$R5 + \frac{47}{6} R3$

b.v.	V	p_1	p_2	p_3	r	s	t	u	value	Row operations
V	1	0	0	0	$\frac{4}{9}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{37}{9}$	$\frac{37}{9}$	$R1+\frac{37}{9}R4$
p_3	0	0	0	1	$\frac{-1}{6}$	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$R2+\frac{3}{2}R4$
p_1	0	1	0	0	$\frac{1}{36}$	$\frac{-1}{6}$	$\frac{5}{36}$	$\frac{4}{9}$	$\frac{4}{9}$	$R3+2R4$
p_2	0	0	1	0	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{-11}{36}$	$\frac{2}{9}$	$\frac{2}{9}$	$R4\div\frac{9}{2}$
P	0	0	0	0	$\frac{4}{9}$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{37}{9}$	$\frac{37}{9}$	$R5+\frac{37}{2}R4$

$V=\frac{37}{9}$ so $v=\frac{37}{9}-5=\frac{-8}{9}$ $p_1=\frac{4}{9}$ $p_2=\frac{2}{9}$ $p_3=\frac{3}{9}$

A should play 1 with probability $\frac{4}{9}$

A should play 2 with probability $\frac{2}{9}$

A should play 3 with probability $\frac{3}{9}$

c

b.v.	V	q_1	q_2	q_3	r	s	t	u	value
r	1	−6	−1	−4	1	0	0	0	0
s	1	−4	−5	−2	0	1	0	0	0
t	1	−1	−7	−5	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	−1	0	0	0	0	0	0	0	0

d

b.v.	V	q_1	q_2	q_3	r	s	t	u	value	Row operations
V	1	−6	−1	−4	1	0	0	0	0	$R1\div 1$
s	0	2	−4	2	−1	1	0	0	0	$R2-R1$
t	0	5	−6	−1	−1	0	1	0	0	$R3-R1$
u	0	1	1	1	0	0	0	1	1	R4 no change
P	0	−6	−1	−4	1	0	0	0	0	$R5+R1$

b.v.	V	q_1	q_2	q_3	r	s	t	u	value	Row operations
V	1	0	$\frac{-41}{5}$	$\frac{-26}{5}$	$\frac{-1}{5}$	0	$\frac{6}{5}$	0	0	R1+6R3
s	0	0	$\frac{-8}{5}$	$\frac{12}{5}$	$\frac{-3}{5}$	1	$\frac{-2}{5}$	0	0	R3−2R3
q_1	0	1	$\frac{-6}{5}$	$\frac{-1}{5}$	$\frac{-1}{5}$	0	$\frac{1}{5}$	0	0	R3÷5
u	0	0	$\frac{11}{5}$	$\frac{6}{5}$	$\frac{1}{5}$	0	$\frac{-1}{5}$	1	1	R4−R3
P	0	0	$\frac{-41}{5}$	$\frac{-26}{5}$	$\frac{-1}{5}$	0	$\frac{6}{5}$	0	0	R5+6R3

b.v.	V	q_1	q_2	q_3	r	s	t	u	value	Row operations
V	1	0	0	$\frac{-8}{11}$	$\frac{6}{11}$	0	$\frac{5}{11}$	$\frac{41}{11}$	$\frac{41}{11}$	$\text{R1}+\frac{41}{5}\text{R4}$
s	0	0	0	$\frac{36}{11}$	$\frac{-5}{11}$	1	$\frac{-6}{11}$	$\frac{8}{11}$	$\frac{8}{11}$	$\text{R2}+\frac{8}{5}\text{R4}$
q_1	0	1	0	$\frac{5}{11}$	$\frac{-1}{11}$	0	$\frac{1}{11}$	$\frac{6}{11}$	$\frac{6}{11}$	$\text{R3}+\frac{6}{5}\text{R4}$
q_2	0	0	1	$\frac{6}{11}$	$\frac{1}{11}$	0	$\frac{-1}{11}$	$\frac{5}{11}$	$\frac{5}{11}$	$\text{R4}\div\frac{11}{5}$
P	0	0	0	$\frac{-8}{11}$	$\frac{6}{11}$	0	$\frac{5}{11}$	$\frac{41}{11}$	$\frac{41}{11}$	$\text{R5}+\frac{41}{5}\text{R4}$

b.v.	V	q_1	q_2	q_3	r	s	t	u	value	Row operations
V	1	0	0	0	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{35}{9}$	$\frac{35}{9}$	$\text{R1}+\frac{8}{11}\text{R2}$
q_3	0	0	0	1	$\frac{-5}{36}$	$\frac{11}{36}$	$\frac{-1}{6}$	$\frac{2}{9}$	$\frac{2}{9}$	$\text{R2}\div\frac{36}{11}$
q_1	0	1	0	0	$\frac{-1}{36}$	$\frac{-5}{36}$	$\frac{1}{6}$	$\frac{4}{9}$	$\frac{4}{9}$	$\text{R3}-\frac{5}{11}\text{R2}$
q_2	0	0	1	0	$\frac{1}{6}$	$\frac{-1}{6}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\text{R4}-\frac{6}{11}\text{R2}$
P	0	0	0	0	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{35}{9}$	$\frac{35}{9}$	$\text{R5}+\frac{8}{11}\text{R2}$

$V=\frac{35}{9}$ so $v=\frac{35}{9}-3=\frac{8}{9}$ $q_1=\frac{4}{9}$ $q_2=\frac{3}{9}$ $q_3=\frac{2}{9}$

B should play 1 with probability $\frac{4}{9}$

B should play 2 with probability $\frac{3}{9}$

B should play 3 with probability $\frac{2}{9}$

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Exercise E, Question 1

Question:

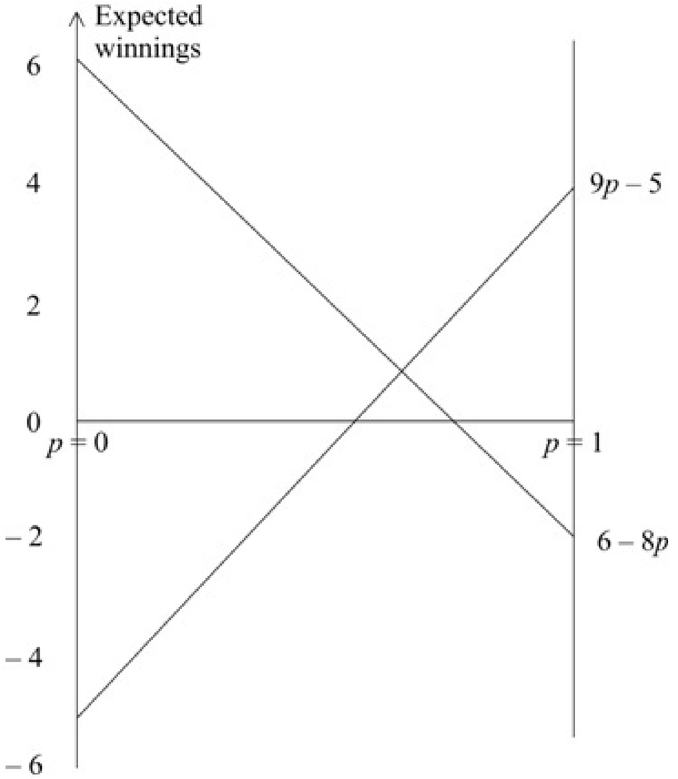
A two-person zero-sum game is represented by the following pay-off matrix for player A. Find the best strategy for each player and the value of the game.

		B	
		I	II
A	I	4	−2
	II	−5	6

Solution:

	B plays 1	B plays 2	Row min	
A plays 1	4	−2	−2	←
A plays 2	−5	6	−5	
Column max	4	6		
	↑			

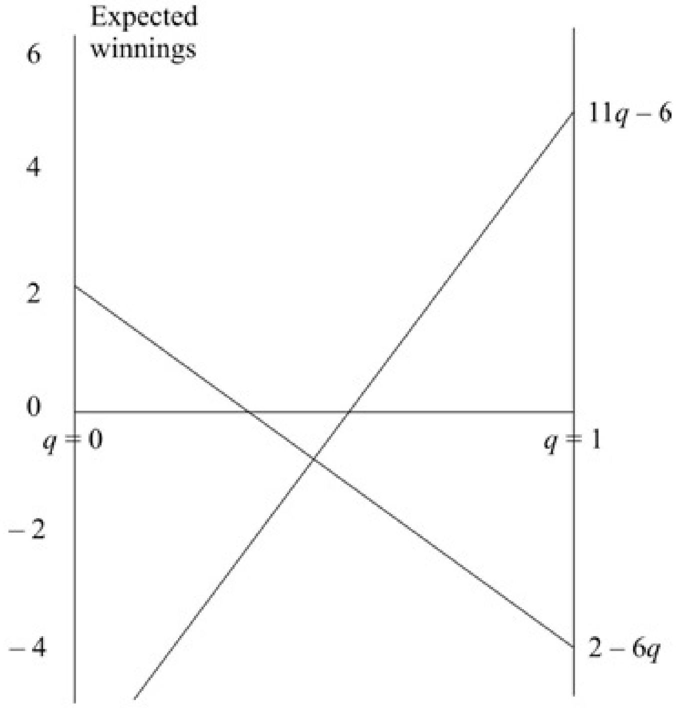
No stable solution since $4 \neq -2$ (column minimax $\neq$ row maximin)
Let A play 1 with probability p
So A plays 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $4p-5(1-p)=9p-5$
If B plays 2 A's expected winnings are $-2p+6(1-p)=6-8p$



$$9p-5=6-8p$$
$$17p=11$$
$$p=\frac{11}{17}$$

A should play 1 with probability $\frac{11}{17}$
A should play 2 with probability $\frac{6}{17}$
The value of the game to A is $\frac{14}{17}$

Let B play 1 with probability q
Let B play 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[4q-2(1-q)]=2-6q$
If A plays 2 B's expected winnings are $-[-5q+6(1-q)]=11q-6$



$$11q - 6 = 2 - 6q$$
$$17q = 8$$
$$q = \frac{8}{17}$$

B should play 1 with probability $\frac{8}{17}$

B should play 2 with probability $\frac{9}{17}$

The value of the game to B is $\frac{-14}{17}$

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Exercise E, Question 2

Question:

Ben and Greg play a zero-sum game, represented by the following pay-off matrix for Ben.

	Greg plays 1	Greg plays 2	Greg plays 3
Ben plays 1	−5	4	3
Ben plays 2	1	−1	−4

a Explain why this matrix might be reduced to

$$\begin{vmatrix} -5 & 3 \\ 1 & -4 \end{vmatrix}$$

b Hence find the best strategy for each player and the value of the game.

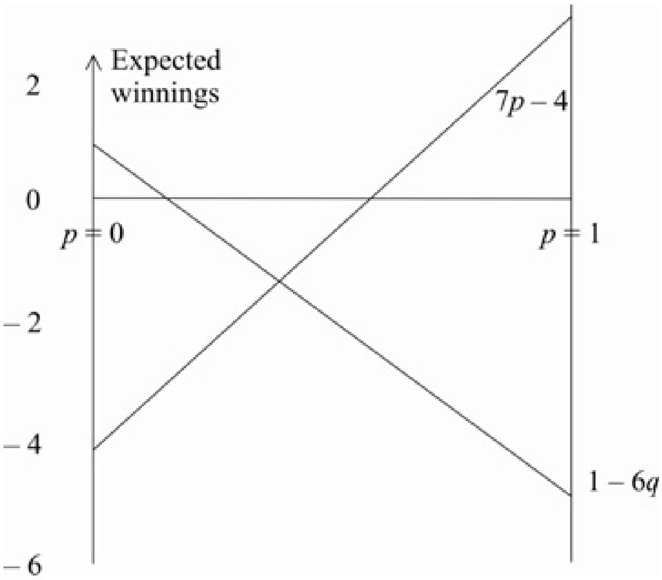
Solution:

a Column 3 dominates column 2 (since $3 < 4$ and $-4 < -1$)

b

	A play 1	A play 2	Row min	
B plays 1	-5	3	-5	
B plays 2	1	-4	-4	←
Col max	1	3		
	↑			

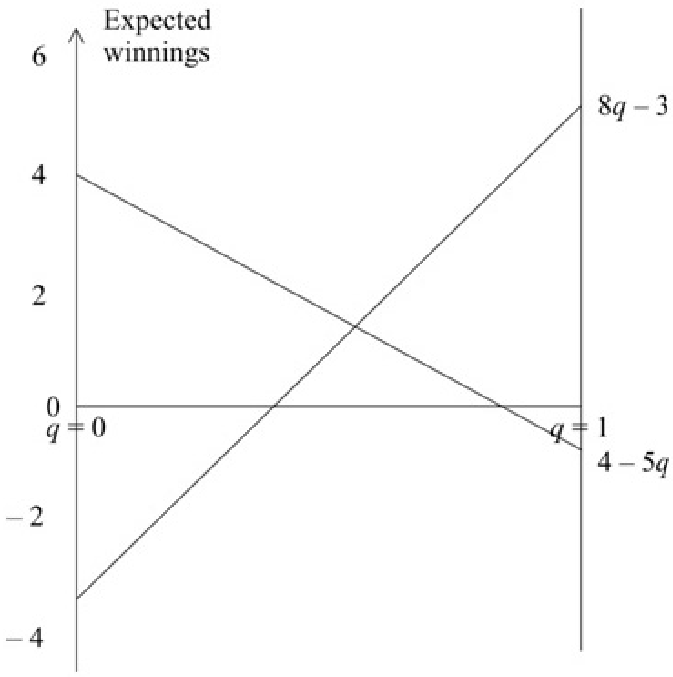
Since $1 \neq -4$ (column minimax $\neq$ row maximin) game is not stable
Let A play 1 with probability p
So A plays 2 with probability $(1-p)$
If B plays 1 A's expected winnings are $-5p+1(1-p)=1-6p$
If B plays 2 A's expected winnings are $3p-4(1-p)=7p-4$



$7p-4=1-6p$
$13p=5$
$p=\frac{5}{13}$

A should play 1 with probability $\frac{5}{13}$
A should play 2 with probability $\frac{8}{13}$
The value of the game is $\frac{-17}{13}$

Let B play 1 with probability q
Let B play 2 with probability $(1-q)$
If A plays 1 B's expected winnings are $-[-5q+3(1-q)]=8q-3$
If A plays 2 B's expected winnings are $-[q-4(1-q)]=4-5q$



$$8q - 3 = 4 - 5q$$
$$13q = 7$$
$$q = \frac{7}{13}$$

B should play 1 with probability $\frac{7}{13}$

B should play 2 with probability $\frac{6}{13}$

The value of the game is $\frac{17}{13}$

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Exercise E, Question 3

Question:

Cait and Georgi play a zero-sum game, represented by the following pay-off matrix for Cait.

$$\begin{array}{r|ccc|} & \text{Georgi plays 1} & \text{Georgi plays 2} & \text{Georgi plays 3} \\ \text{Cait plays 1} & -5 & 2 & 3 \\ \text{Cait plays 2} & 1 & -3 & -4 \\ \text{Cait plays 3} & -7 & 0 & 1 \end{array}$$

a Identify the play safe strategies for each player.
b Verify that there is no stable solution to this game.
c Use dominance to reduce the game to a 2×3 game, explaining your reasoning.
d Find Cait's best strategy and the value of the game to her.
e Write down the value of the game to Georgi.

Solution:

	G plays 1	G plays 2	G plays 3	Row min	
C plays 1	−5	2	3	−5	
C plays 2	1	−3	−4	−4	←
C plays 3	−7	0	1	−7	
Column max	1	2	3		
	↑				

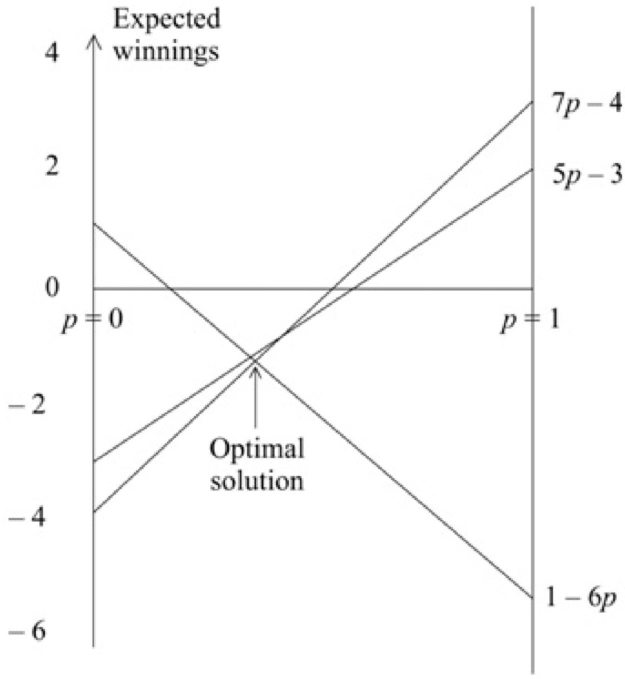
a Play safe: Cait plays 2 Georgi plays 1

b $1 \neq -4$ (column minimax ≠ row maximin) so no stable solution

c Row 1 dominates row 3 (since $-5 > -7 \quad 2 > 0 \quad 3 > 1$)

	G plays 1	G plays 2	G plays 3
C plays 1	−5	2	3
C plays 2	1	−3	−4

d Let C play 1 with probability p
So C plays 2 with probability $(1-p)$
If G plays 1 C's expected winnings are $-5p+1(1-p)=1-6p$
If G plays 2 C's expected winnings are $2p-3(1-p)=5p-3$
If G plays 3 C's expected winnings are $3p-4(1-p)=7p-4$



$$7p-4=1-6p$$
$$13p=5$$
$$p=\frac{5}{13}$$

Cait should play 1 with probability $\frac{5}{13}$
Cait should play 2 with probability $\frac{8}{13}$
Cait should play 3 never
The value of the game is $\frac{-17}{13}$

e The value of the game to Georgi is $\frac{17}{13}$

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Exercise E, Question 4

Question:

A two person zero-sum game is represented by the following pay-off matrix for player A.

	B plays 1	B plays 2	B plays 3
A plays 1	2	−1	−3
A plays 2	−2	1	4
A plays 3	−3	1	−3
A plays 4	−1	2	−2

a Verify that there is no stable solution to this game

b Explain the circumstances under which a row, x, dominates a row, y.

c Reduce the game to a 3×3 game, explaining your reasoning.

d Formulate the 3×3 game as a linear programming problem for player A. Write the constraints as inequalities and define your variables.

Solution:

a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	2	−1	−3	−3	
A plays 2	−2	1	4	−2	←
A plays 3	−3	1	−3	−3	
A plays 4	−1	2	−2	−2	←
Column max	2	2	4		
	↑	↑			

Since $2 \neq -2$ (column minimax $\neq$ row maximin) there is no stable solution.

b A row x dominates a row y, if, in each column, the element in row $x \geq$ the element in row y.

c Row 4 dominates row 3

	B plays 1	B plays 2	B plays 3
A plays 1	2	−1	−3
A plays 2	−2	1	4
A plays 3	−1	2	−2

d Add 4 to all elements

	B plays 1	B plays 2	B plays 3
A plays 1	6	3	1
A plays 2	2	5	8
A plays 3	3	6	2

Let A play 1 with probability p_1
Let A play 2 with probability p_2
Let A play 3 with probability p_3
Let the value of the game to A be v so $V = v + 4$
Maximise $P = V$
Subject to:

$$6p_1 + 2p_2 + 3p_3 \geq V$$
$$3p_1 + 5p_2 + 6p_2 \geq V$$
$$p_1 + 8p_2 + 2p_3 \geq V$$
$$p_1 + p_2 + p_3 \leq 1$$

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Exercise E, Question 5

Question:

A two person zero-sum game is represented by the following pay-off matrix for player A.

$$\begin{array}{c|ccc|}
 & \text{B plays 1} & \text{B plays 2} & \text{B plays 3} \\
\text{A plays 1} & 5 & -1 & 1 \\
\text{A plays 2} & -1 & -4 & 4 \\
\text{A plays 3} & 3 & -2 & -1
\end{array}$$

a Identify the play safe strategies for each player.
b Verify that there is no stable solution to this game.
c Use dominance to reduce the game to a 3×2 game, explaining your reasoning.
d Write down the pay-off matrix for player B.
e Find B's best strategy and the value of the game.

Solution:

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	5	−3	1	−3	
A plays 2	−1	−4	4	−4	
A plays 3	3	2	−1	−1	←
Column max	5	2	4		
		↑			

a Play safe (A plays 1, B plays 2)

b Since $2 \neq -1$ (column minimax $\neq$ row maximin) there is no stable solution

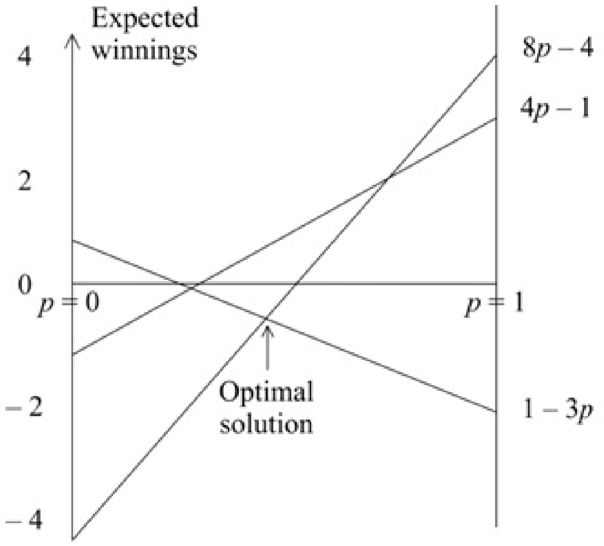
c Column 2 dominates column 1 $(-3 < 5, -4 < -1, 2 < 3)$ B would always choose to minimise A's winnings by playing 2 rather than 1

	B plays 2	B plays 3
A plays 1	−3	1
A plays 2	−4	4
A plays 3	2	−1

d

	A plays 1	A plays 2	A plays 3
B plays 2	3	4	−2
B plays 3	−1	−4	1

e Let B play 2 with probability p
So B plays 3 with probability $(1-p)$
If A plays 1 B's expected winnings are $3p-1(1-p)=4p-1$
If A plays 2 B's expected winnings are $4p-4(1-p)=8p-4$
If A plays 3 B's expected winnings are $-2p+1(1-p)=1-3p$



$$8p-4=1-3p$$
$$11p=5$$
$$p=\frac{5}{11}$$

B should play 2 with probability $\frac{5}{11}$

B should play 3 with probability $\frac{6}{11}$

The value of the game is $\frac{-4}{11}$

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Exercise E, Question 6

Question:

A two person zero-sum game is represented by the following pay-off matrix for player A.

$$\begin{array}{c|ccc|} & \text{B plays1} & \text{B plays2} & \text{B plays3} \\ \text{A plays1} & 2 & 7 & -1 \\ \text{A plays2} & 5 & 0 & 8 \\ \text{A plays3} & -2 & 3 & 5 \end{array}$$

a Identify the play safe strategies for each player.
b Verify that there is no stable solution to this game.
c Write down the pay-off matrix for player B
d Formulate the game for player B as a linear programming problem. Define your variables and write your constraints as equations.
e Write down an initial tableau that you could use to solve the game for player B.

Solution:

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	2	7	−1	−1	
A plays 2	5	0	8	0	←
A plays 3	−2	3	5	−2	
Column max	5	7	8		
	↑				

a Play safe is (A plays 2, B plays 1)

b Since $5 \neq 0$ (column minimax $\neq$ row maximin) there is no stable solution

c

	A plays 1	A plays 2	A plays 3
B plays 1	−2	−5	2
B plays 2	−7	0	−3
B plays 3	1	−8	−5

d Adding 9 to all elements

	A plays 1	A plays 2	A plays 3
B plays 1	7	4	11
B plays 2	2	9	6
B plays 3	10	1	4

Let B play 1 with probability p_1, play 2 with probability p_2 and play 3 with probability p_3.

Let v = value of the game to B and $V = v + 9$

Maximise $P = V$

Subject to:

$$7p_1 + 2p_2 + 10p_3 \geq V \Rightarrow V - 7p_1 - 2p_2 - 10p_3 + r = 0$$
$$4p_1 + 9p_2 + p_3 \geq V \Rightarrow V - 4p_1 - 9p_2 - p_3 + s = 0$$
$$11p_1 + 6p_2 + 4p_3 \geq V \Rightarrow V - 11p_1 - 6p_2 - 4p_3 + t = 0$$
$$p_1 + p_2 + p_3 \leq 1 \Rightarrow p_1 + p_2 + p_3 + u = 1$$
$$\text{where } p_1, p_2, p_3, r, s, t, u \geq 0$$

e

b.v.	V	P_1	P_2	P_3	r	s	t	u	value
r	1	−7	−2	−10	1	0	0	0	0
s	1	−4	−9	−1	0	1	0	0	0
t	1	−11	−6	−4	0	0	1	0	0
u	0	1	1	1	0	0	0	1	1
P	−1	0	0	0	0	0	0	0	0

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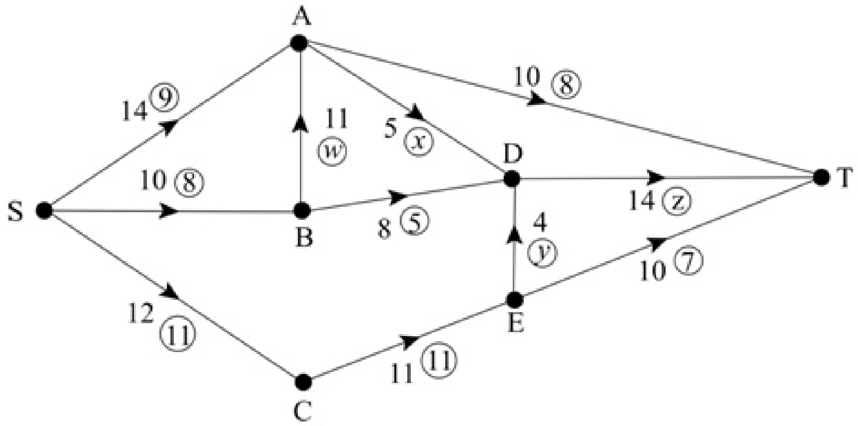
Exercise A, Question 1

Question:



The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow pattern.

a Find the values of w, x, y and z, explaining your reasoning.

b State the value of the initial flow.

c Identify two saturated arcs.

d Write down the capacity of arc BD.

e What is the current flow along route SAT?

Solution:

a Flow into B = flow out of B $\quad w = 3$

Flow into A = flow out of A $\quad x = 4$

Flow into E = flow out of E $\quad y = 4$

Flow into D = flow out of D $\quad z = 13$

b Feasible flow = 28

c CE and ED are saturated

d BD has capacity 8

e Along SAT the current flow is 8

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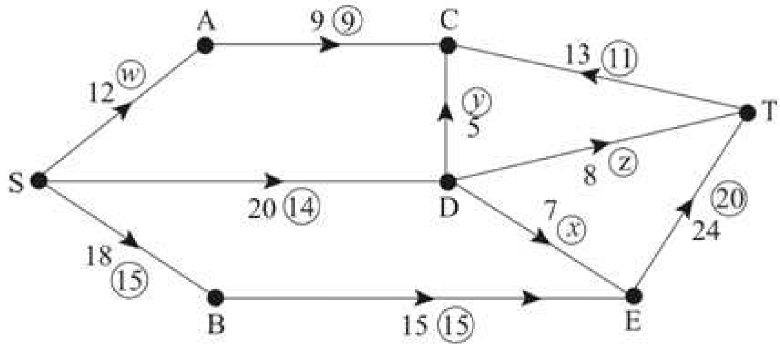
Exercise A, Question 2

Question:



The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow pattern.

a Find the values of w, x, y and z, explaining your reasoning.
b State the value of the initial flow.
c Identify two saturated arcs.
d Write down the flow along arc SD.
e What is the current flow along the route SBET?

Solution:

a Flow into A = flow out of A $\quad w = 9$
Flow into E = flow out of E $\quad x = 5$
Flow into C = flow out of C $\quad y = 2$
Flow into D = flow out of D $\quad 14 = y + x + 2 \Rightarrow 14 = 2 + 5 + z \Rightarrow z = 7$

b Feasible flow = 38

c BE and AC are saturated

d Flow along SD is 14

e Flow along SBET = 15

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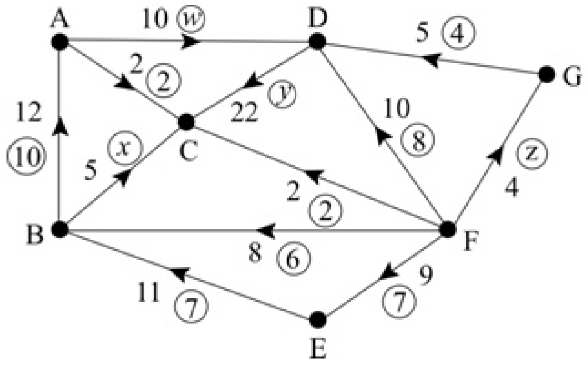
Exercise A, Question 3

Question:



The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow pattern.

a State the source vertex.
b State the sink vertex.
c Find the values of w, x, y and z, explaining your reasoning.
d State the value of the feasible flow.
e Identify three saturated arcs.
f Write down the capacity of arc FB.

Solution:

a Source vertex is F

b Sink vertex is C

c Flow into A = flow out of A $w = 8$
Flow into B = flow out of B $x = 3$
Flow into D = flow out of D $y = 20$
Flow into G = flow out of G $z = 4$

d Feasible flow = 27

e Saturated arcs are AC, FC, FG

f Capacity of FB is 8

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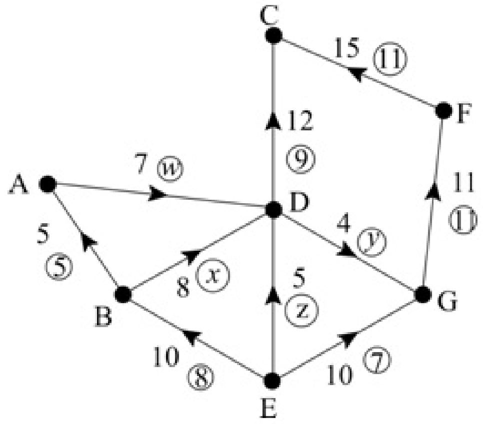
Exercise A, Question 4

Question:



The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. The numbers in circles represent an initial flow pattern.

a State the source vertex.
b State the sink vertex.
c Find the values of w, x, y and z, explaining your reasoning.
d State the value of the initial flow.
e Identify four saturated arcs.
f Write down the flow along arc FC.

Solution:

a Source vertex is E
b Sink vertex is C
c Flow into A = flow out of A $w = 5$
Flow into B = flow out of B $x = 3$
Flow into G = flow out of G $y = 4$
Flow into D = flow out of D $z = 5$

d Feasible flow = 20

e Saturated arcs are BA, ED, DG, GF

f Flow along FC = 11

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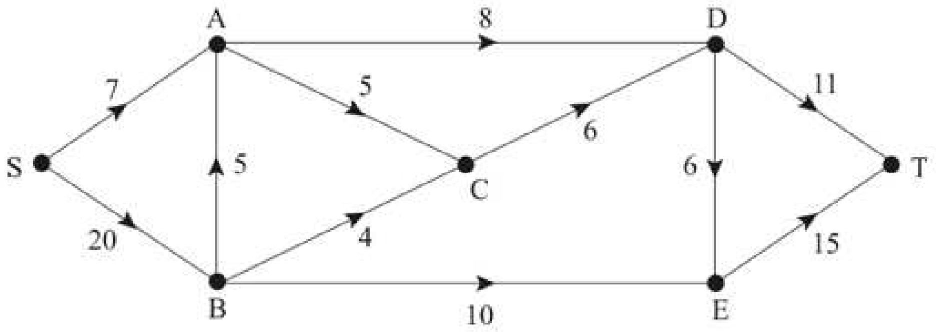
Exercise A, Question 5

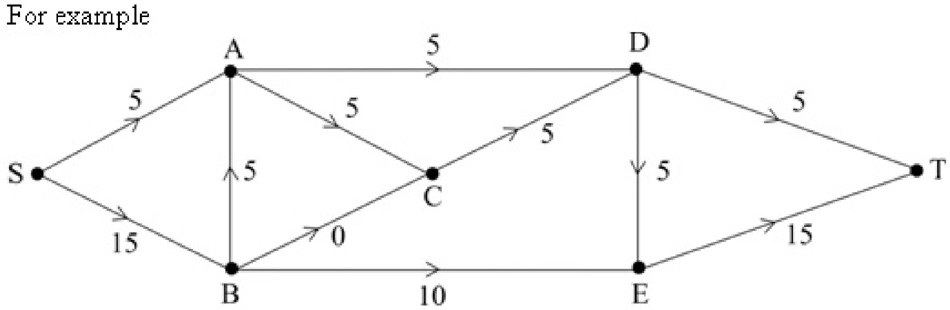
Question:



The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. Find a feasible flow of at least 20 through the network from S to T.

Solution:

For example



There are many other answers.

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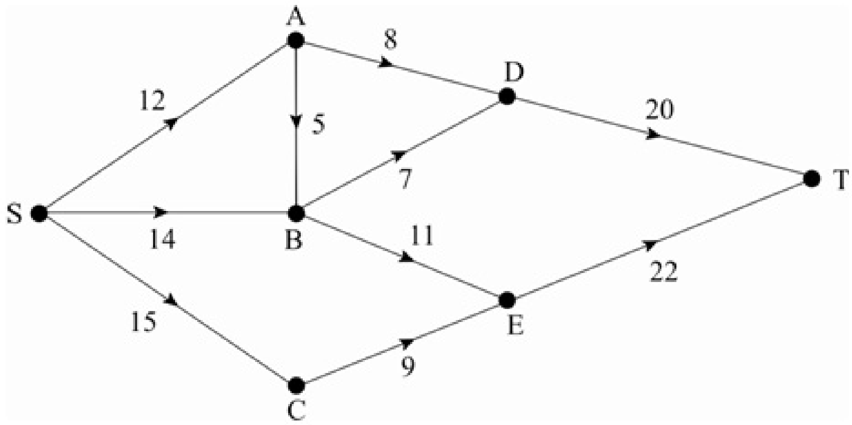
Exercise A, Question 6

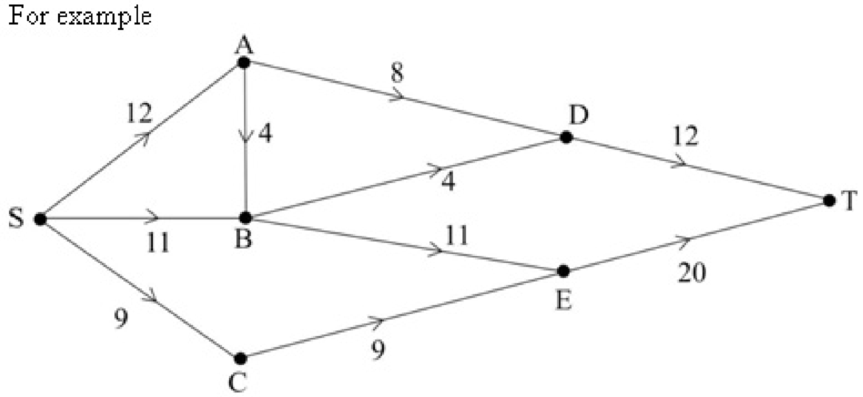
Question:



The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. Find a feasible flow of 32 through the network from S to T.

Solution:

For example



There are many other solutions.

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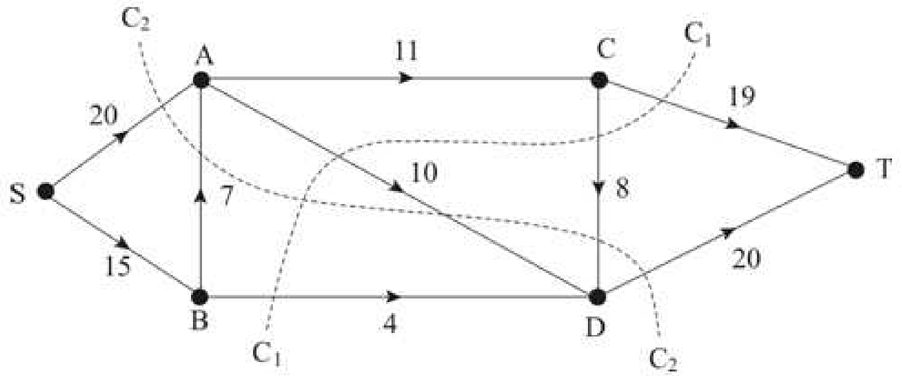
Exercise B, Question 1

Question:

The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. Where relevant, the numbers in circles represent an initial flow pattern. Evaluate the capacities of the cuts drawn.



Solution:

Cut $C_1 = 19 + 8 + 10 + 4 = 41$
Cut $C_2 = 20 + 7 + 20 = 47$

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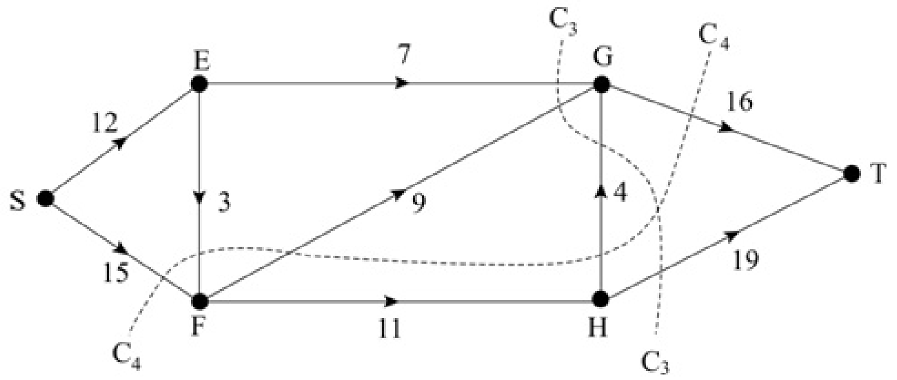
Exercise B, Question 2

Question:

The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. Where relevant, the numbers in circles represent an initial flow pattern. Evaluate the capacities of the cuts drawn.



Solution:

Cut $C_3 = 7 + 9 + 4 + 19 = 39$
Cut $C_4 = 15 + 3 + 16 = 34$

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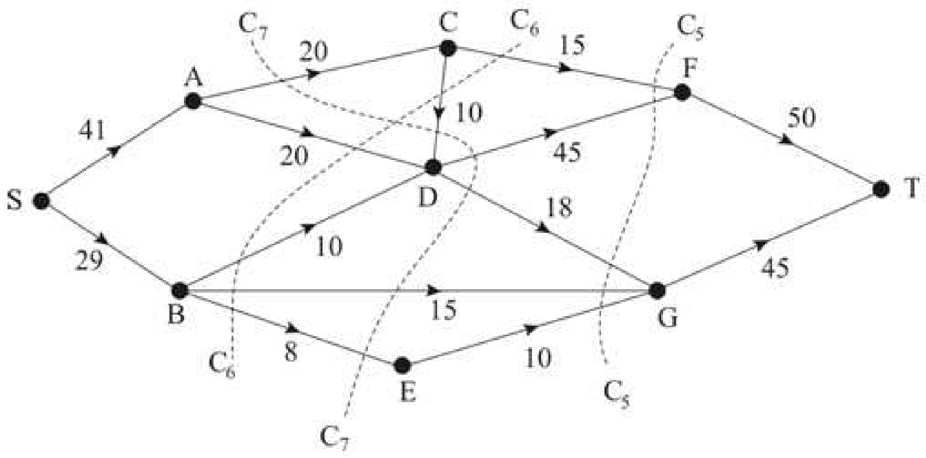
Exercise B, Question 3

Question:

The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. Where relevant, the numbers in circles represent an initial flow pattern. Evaluate the capacities of the cuts drawn.



Solution:

Cut $C_5 = 15+45+18+15+10=103$

Cut $C_6 = 15+10+20+10+15+8=78$

Cut $C_7 = 20+45+18+15+8=106$

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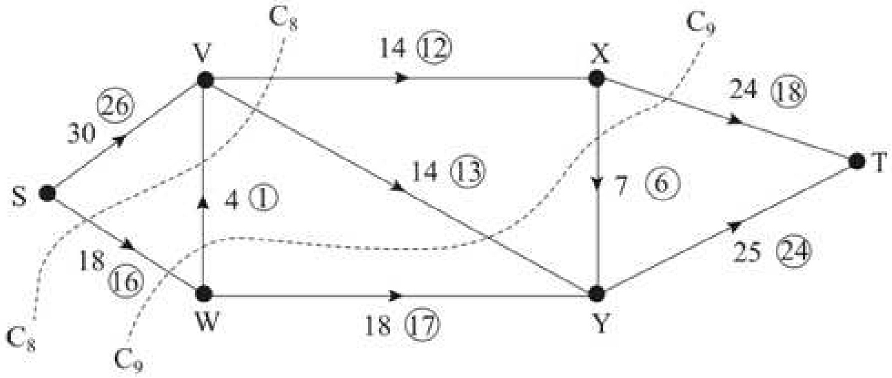
Exercise B, Question 4

Question:

The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. Where relevant, the numbers in circles represent an initial flow pattern. Evaluate the capacities of the cuts drawn.



Solution:

Cut $C_8 = 14+14+4+18 = 50$
Cut $C_9 = 24+14+4+18 = 60$

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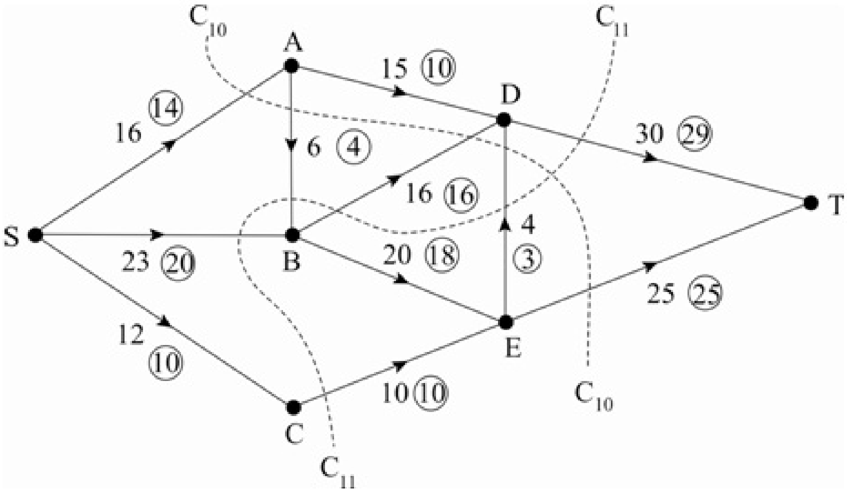
Exercise B, Question 5

Question:

The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. Where relevant, the numbers in circles represent an initial flow pattern. Evaluate the capacities of the cuts drawn.



Solution:

Cut $C_{10} = 16 + 16 + 4 + 25 = 61$
Cut $C_{11} = 30 + 6 + 23 + 10 = 69$

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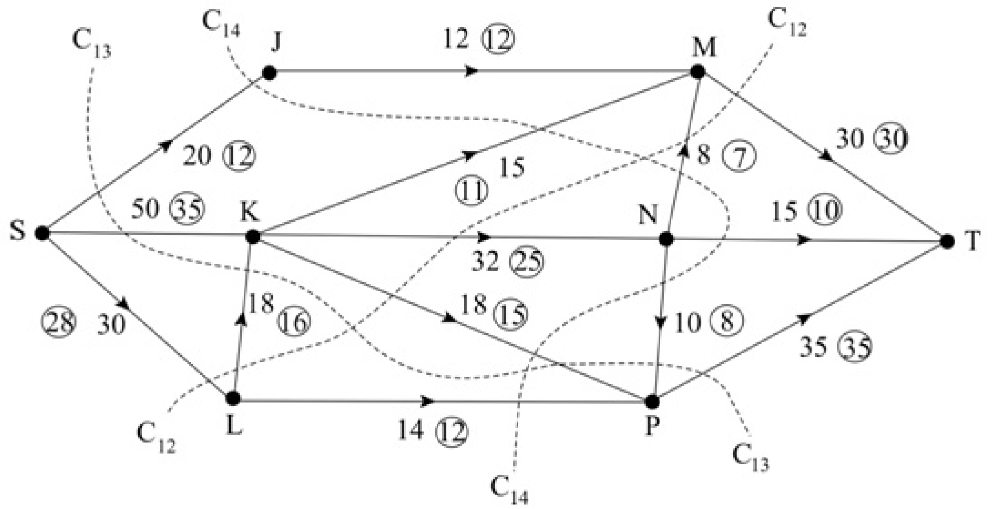
Exercise B, Question 6

Question:

The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc. Where relevant, the numbers in circles represent an initial flow pattern. Evaluate the capacities of the cuts drawn.



Solution:

Cut $C_{12} = 30 + 32 + 18 + 30 = 110$

Cut $C_{13} = 20 + 50 + 18 + 35 = 123$

Cut $C_{14} = 20 + 15 + 8 + 15 + 10 + 18 + 14 = 100$

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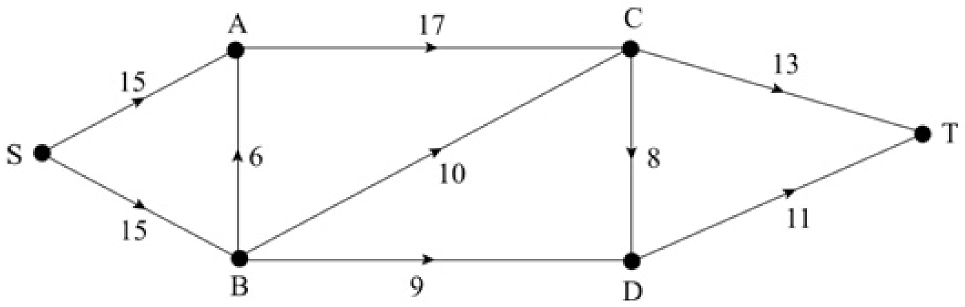
Exercise C, Question 1

Question:



The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc.

a State the maximum flows along SACT and SBCDT.

b Show these on a diagram.

Using this as your initial flow,

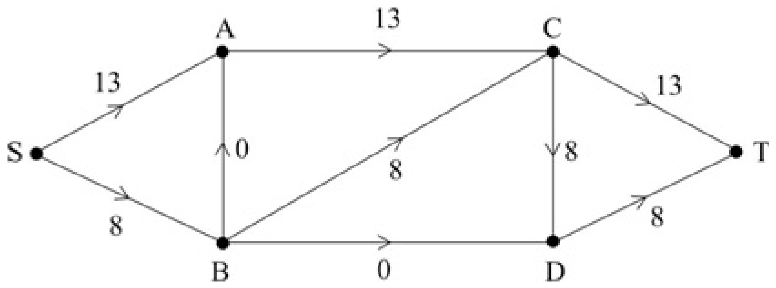
c calculate the value of the initial flow.

Solution:

a max flow along SACT = 13

max flow along SBCDT = 8

b



c Value of initial flow = 21

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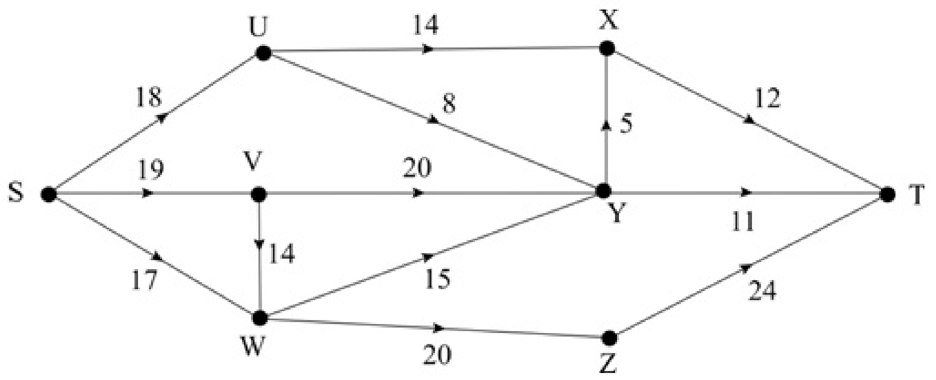
Exercise C, Question 2

Question:



The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc.

a State the maximum flows along SUXT, SWZT and SVWYT.

b Show these on a diagram.

Using this as your initial flow,

c calculate the value of the initial flow.

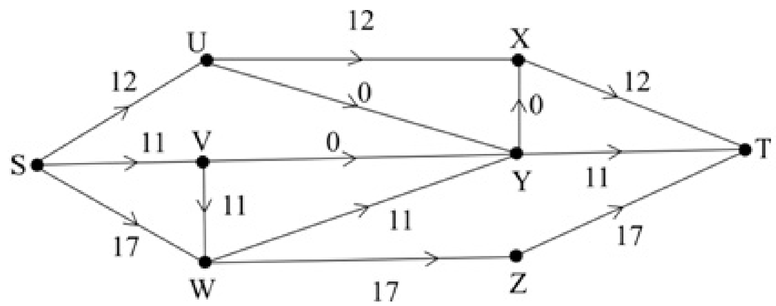
Solution:

a max flow along SUXT = 12

max flow along SWZT = 17

max flow along SVWYT = 11

b



c Value of initial flow = 40

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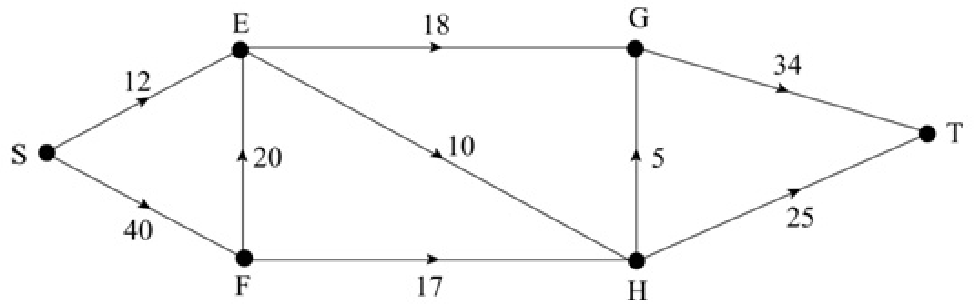
Exercise C, Question 3

Question:



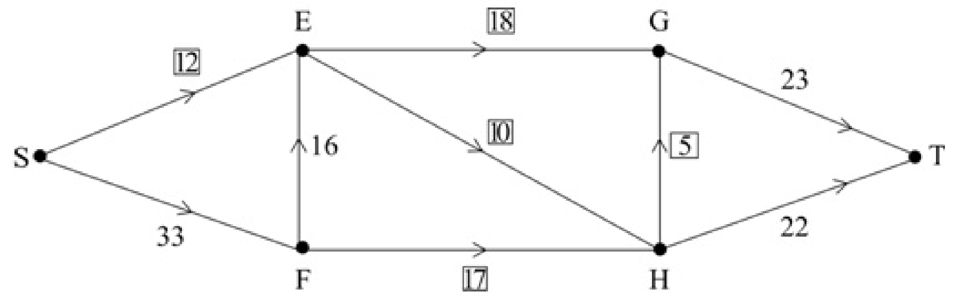
The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc.

a Given that arcs SE, EG, EH, FH and HG are saturated, draw an initial flow through the network.

b State the value of the initial flow.

Solution:

a



b Value of initial flow = 45

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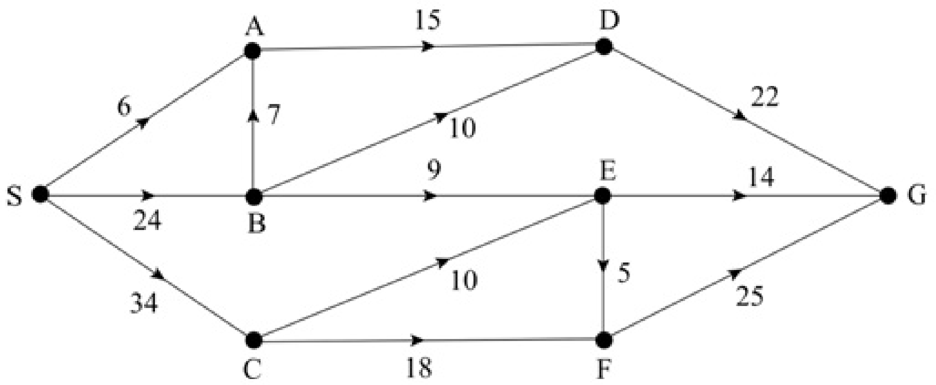
Exercise C, Question 4

Question:



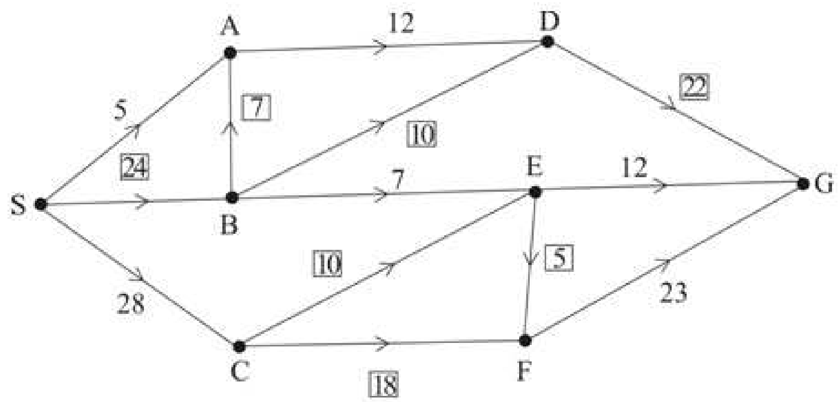
The diagram shows a capacitated directed network. The number on each arc represents the capacity of that arc.

a Given that arcs SB, BA, BD, CE, CF, EF and DG are saturated, draw an initial flow through the network.

b State the value of the initial flow.

Solution:

a



b Value of initial flow 57

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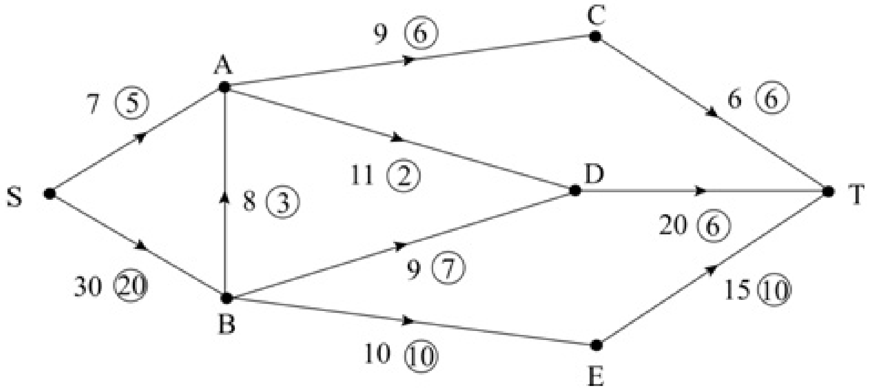
Exercise D, Question 1

Question:

The diagram shows a capacitated, directed network. The capacity of each arc is shown on each arc. The numbers in circles represent an initial flow from S to T.

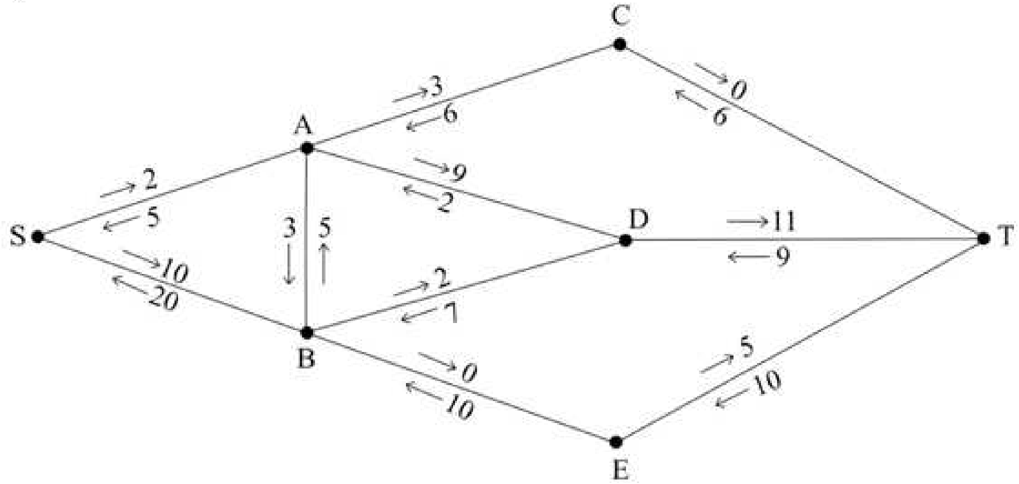
a Starting from the initial flow, use the labelling procedure to find a maximum flow through the network. You must list each flow-augmenting route you use together with its flow.

b Draw your final flow pattern and state the value of your maximum flow.



Solution:

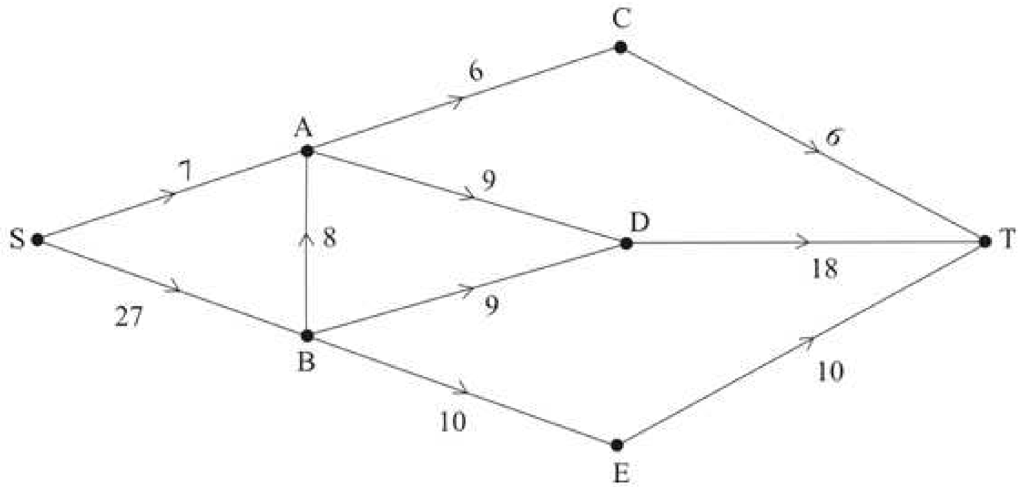
a



For example (there are many other combinations of flows possible)

SBADT – 5
SADT – 2
SBDT – 2

b



Value of maximum flow is 34

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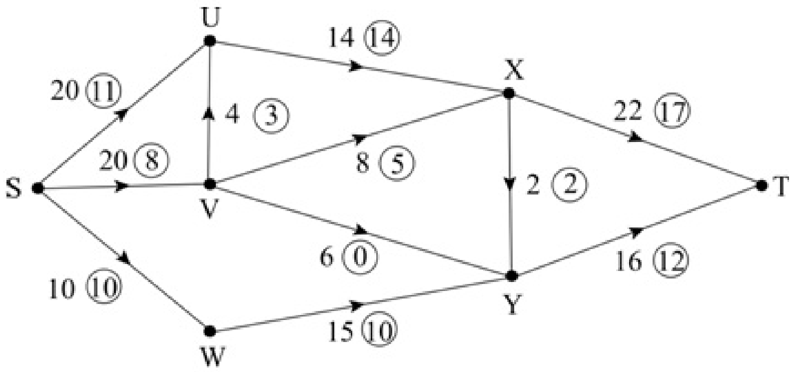
Exercise D, Question 2

Question:

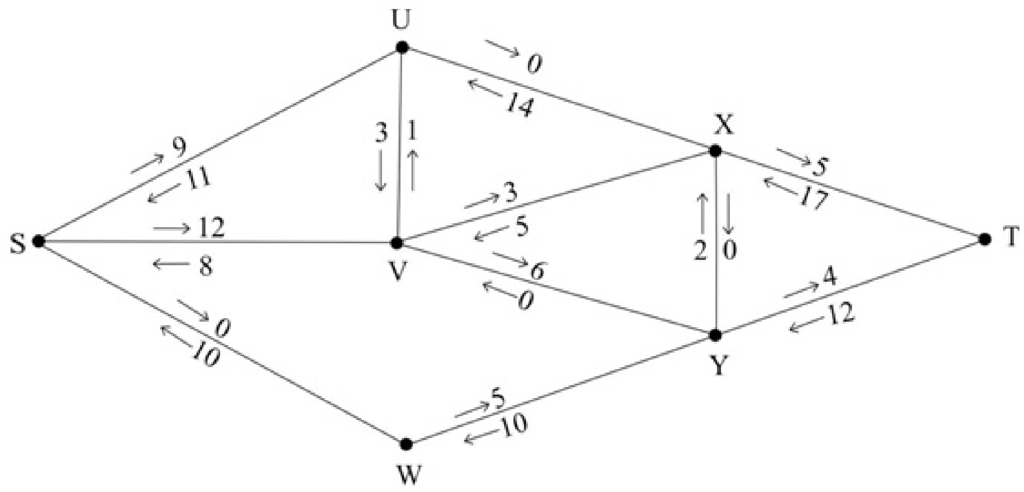
The diagram shows a capacitated, directed networks. The capacity of each arc is shown on each arc. The numbers in circles represent an initial flow from S to T.

a Starting from the initial flow, use the labelling procedure to find a maximum flow through the network. You must list each flow-augmenting route you use together with its flow.

b Draw your final flow pattern and state the value of your maximum flow.



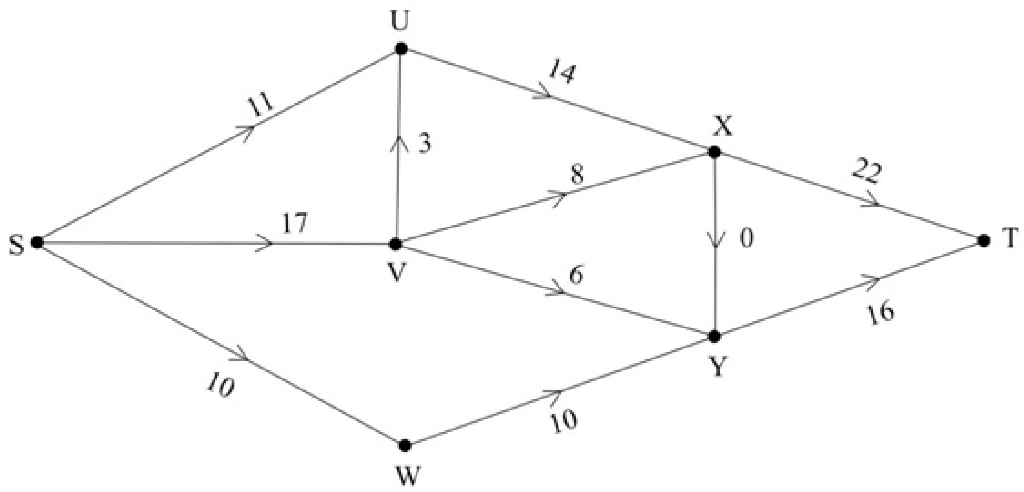
Solution:



For example(there are many other combinations of flows possible)

SVYT – 4
SVXT – 3
SVYXT – 2

b



Value of maximum flow is 38

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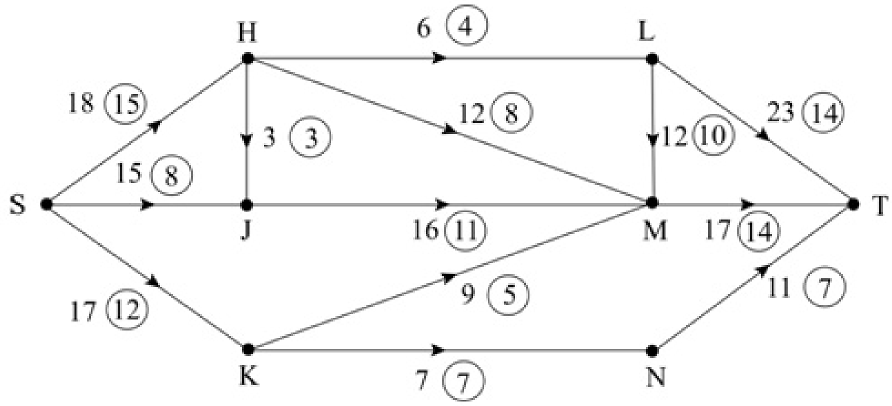
Exercise D, Question 3

Question:

The diagram shows a capacitated, directed networks. The capacity of each arc is shown on each arc. The numbers in circles represent an initial flow from S to T.

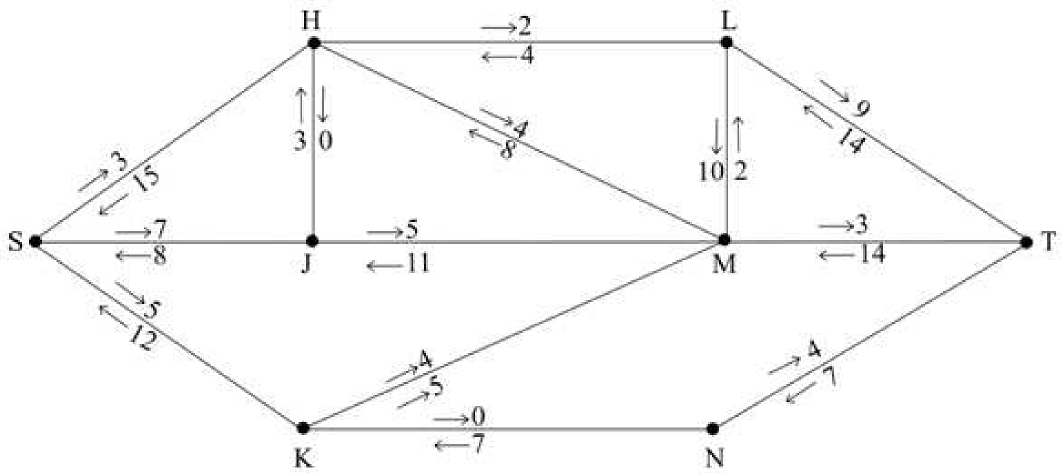
a Starting from the initial flow, use the labelling procedure to find a maximum flow through the network. You must list each flow-augmenting route you use together with its flow.

b Draw your final flow pattern and state the value of your maximum flow



Solution:

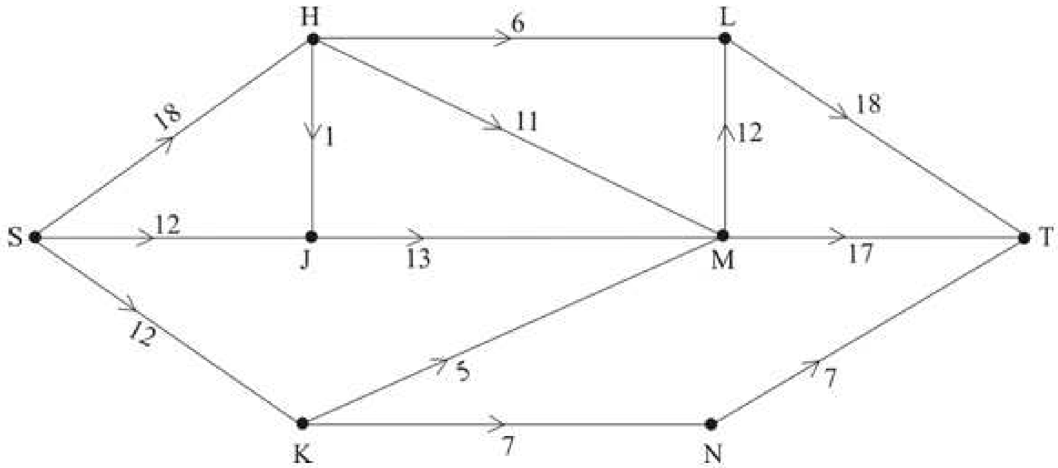
a



For example (there are many other combinations of flows)

SHMT – 3
SJMLT – 2
SJHLT – 2

b



Value of maximum flow is 42

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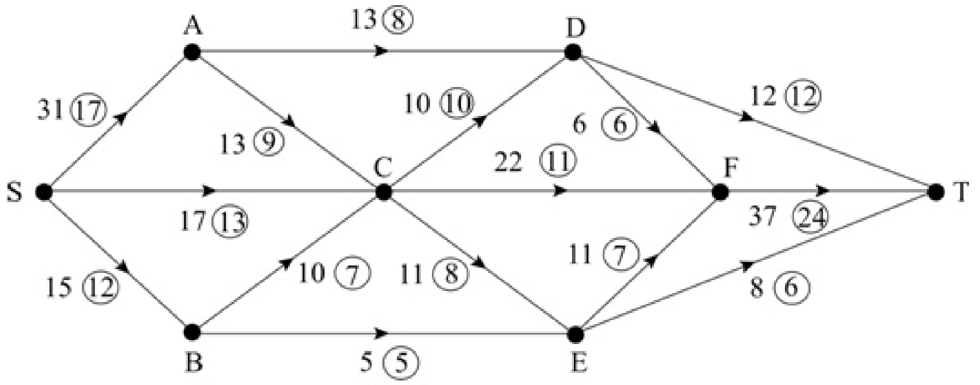
Exercise D, Question 4

Question:

The diagram shows a capacitated, directed networks. The capacity of each arc is shown on each arc. The numbers in circles represent an initial flow from S to T.

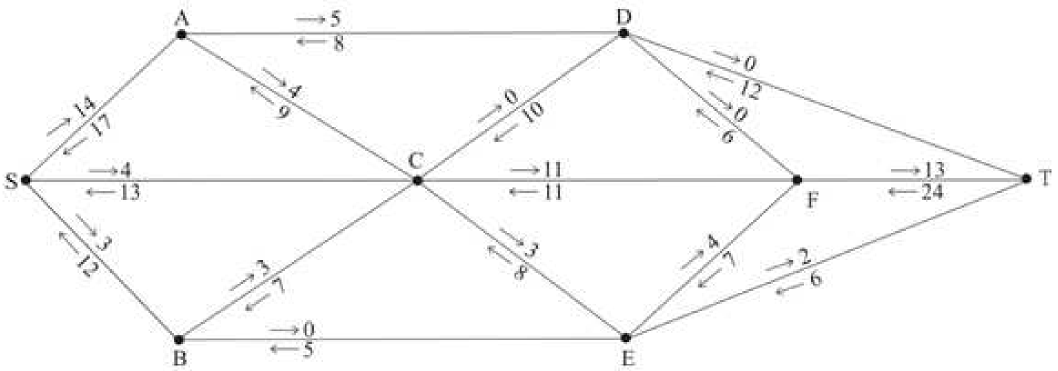
a Starting from the initial flow, use the labelling procedure to find a maximum flow through the network. You must list each flow-augmenting route you use together with its flow.

b Draw your final flow pattern and state the value of your maximum flow



Solution:

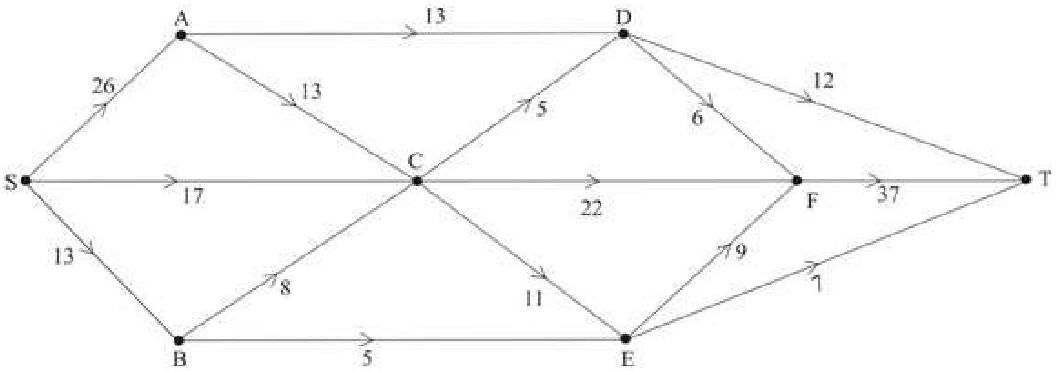
a



For example (there are many other combinations of flow)

SACFT – 4
SADCFT – 5
SCFT – 2
SCEFT – 2
SBCET – 1

b



Value of maximum flow is 56

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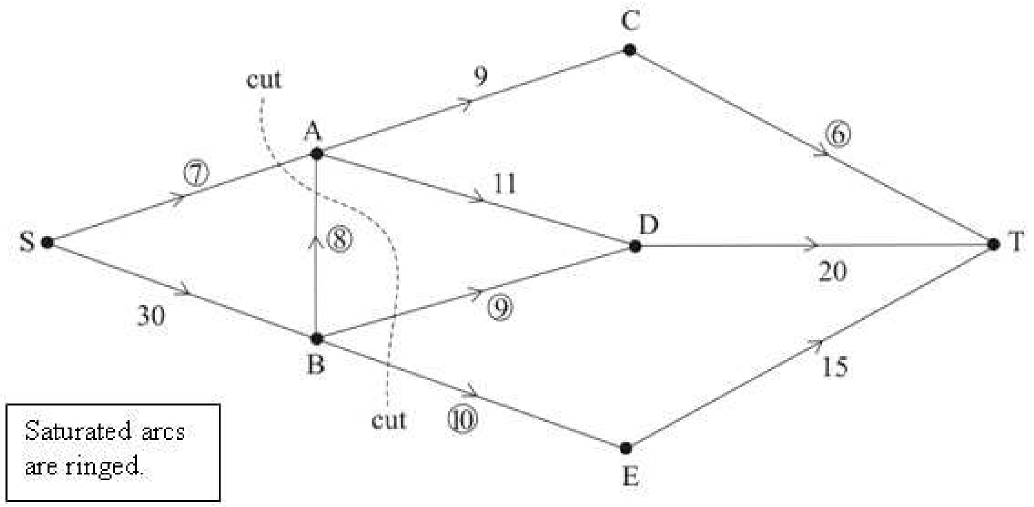
Exercise E, Question 1

Question:

Use the maximum flow-minimum cut theorem to prove that the flows you found in answer to the questions in exercise 6D are maximal.

Solution:



The diagram shows the capacity of each arc.
So by maximum flow − minimum cut theorem flow is maximal

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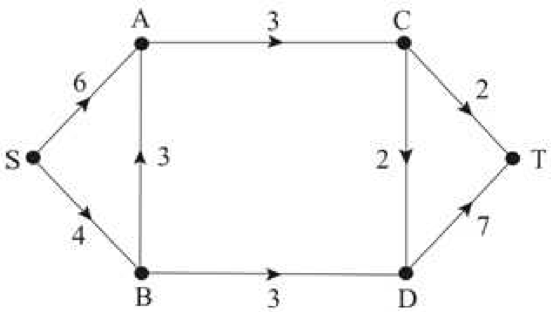
Exercise F, Question 1

Question:



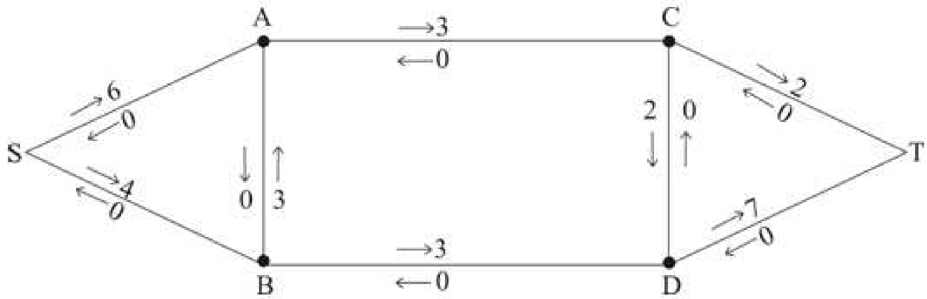
The diagram shows a capacitated, directed network. The number on each arc indicates the capacity of that arc.

a Use the labelling procedure to find the maximum flow through the network from S to T, listing each flow augmenting route you use, together with its flow.

b Verify that the flow found in part **a** is maximal.

Solution:

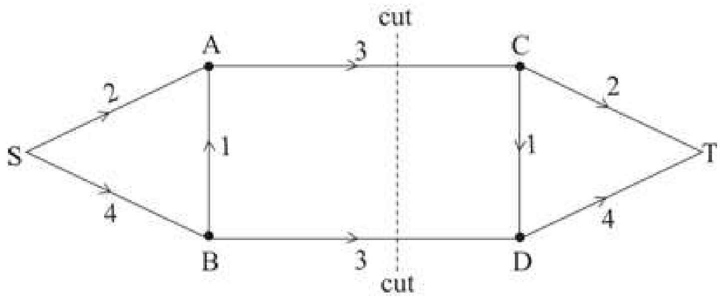
a



For example (many different flow combinations are possible)

SBDT − 3
SACT − 2
SBACDT − 1

b



Minimum cut = 6 so by maximum flow = minimum cut theorem, flow is maximum.

Solutionbank D2

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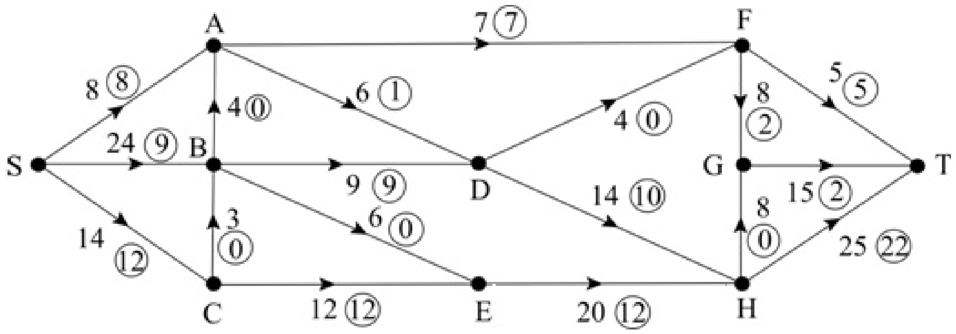
Exercise F, Question 2

Question:



The diagram shows a capacitated directed network. The number on each arc is the value of the maximum flow along that arc.

a Describe briefly a situation for which this type of network could be a suitable model.

The numbers in circles show a feasible flow of value 29 from source S to sink T. Take this as the initial flow pattern.

b Use the labelling procedure to find the maximum flow through the network from S to T. You must list each flow-augmenting route you use together with its flow.

c Indicate your maximum flow pattern and state the final flow.

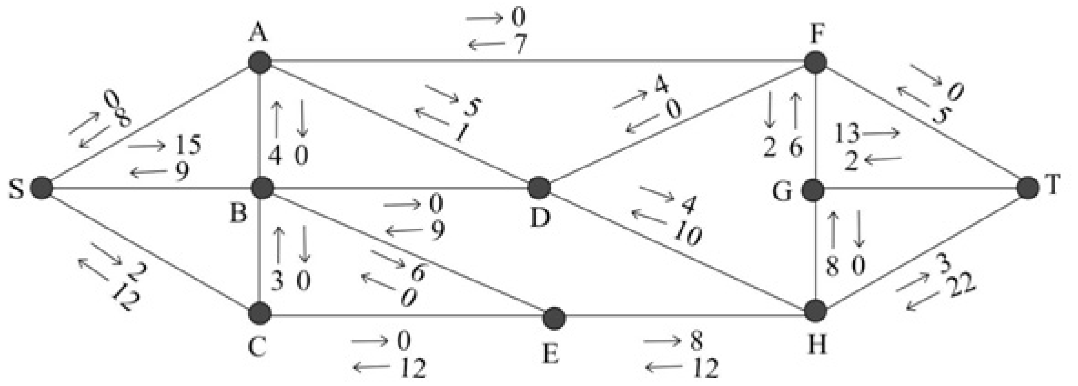
d Verify that your answer is a maximum flow by using the maximum flow–minimum cut theorem, listing the arcs through which your cut passes.

e For the maximum flow, state a property of the arcs found in **d**. **[*E*]**

Solution:

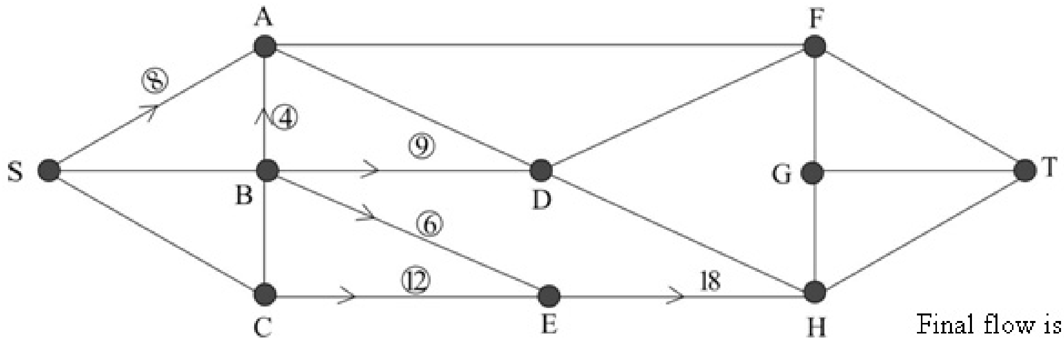
a Applied. Idea of flow through a system, idea of directed flow.
e.g. traffic moving through a one-way system of roads

b



e.g. S B E H G T – 6 and S B A D F G T – 4
or S B A D H G T – 4 and S C B E H T – 2 and S B E H G T – 4 etc.

c Many solutions are possible, but the following values must be given. Flows must be consistent.



Final flow is 39

d Cut through SA, BA, BD, BE and CE

e The arcs are saturated.

Solutionbank D2

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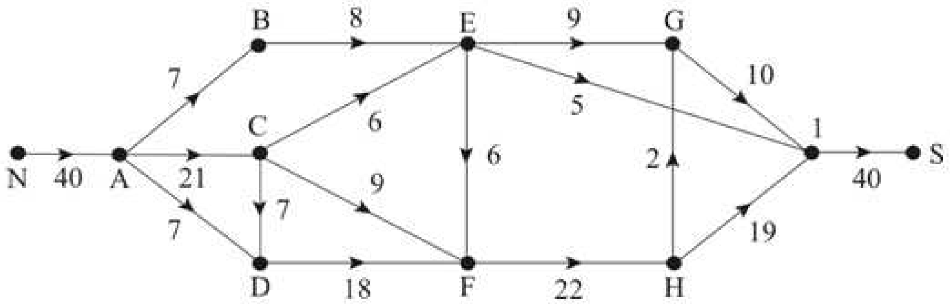
Exercise F, Question 3

Question:



The diagram shows the road routes from a bus station, N, on the north side of a town to a bus station S, on its south side. The number on each arc shows the maximum flow rate, in vehicles per minute, on that route.

a State four junctions at which there could be traffic delays, giving a reason for your answer.

Given that AB, AD, CE, CF and EI are saturated,

b show a flow of 31 from N to S that satisfies this demand.

c Taking your answer to **b** as the initial flow pattern, use the labelling procedure to find the maximum flow. You should list each flow-augmenting route you use together with its flow.

d Indicate your maximum flow pattern.

e Verify your solution using the maximum flow–minimum cut theorem, listing the arcs through which your minimum cut passes.

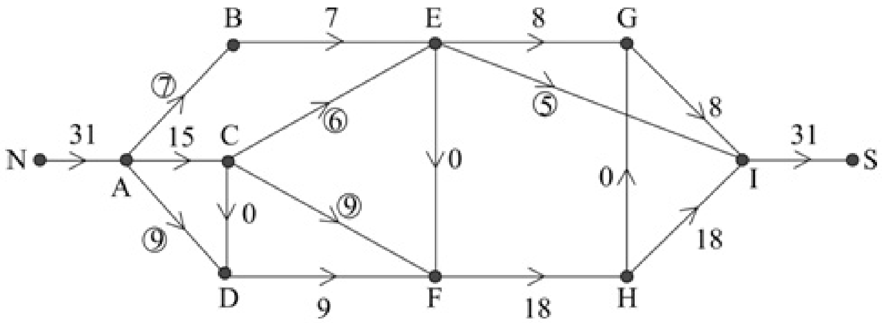
f Show that, in this case, there is a second minimum cut and list the arcs through which it passes.

[*E*]

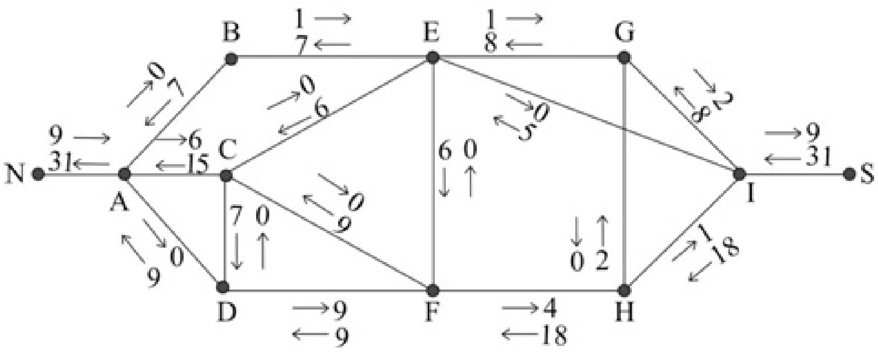
Solution:

a A F G and H, possible flow in > possible flow out.

b e.g.



c Using labelling procedure e.g.

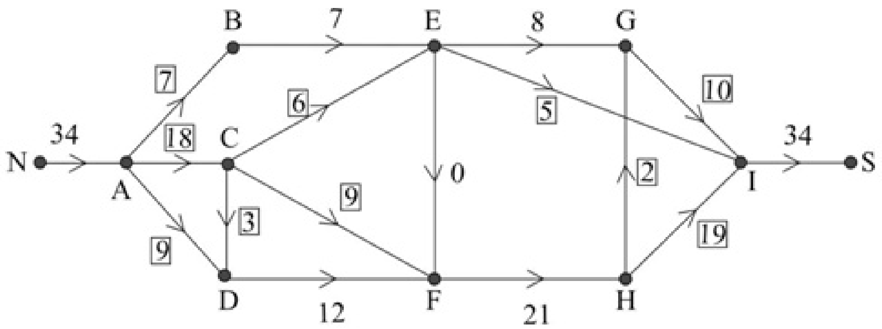


e.g.

If H I = 16	If HI = 17	If HI = 18	If H I = 19
NACDFHIS – 3	NACDFHGIS – 1	NACDFHGIS – 2	NACDFHGIS – 1
	NACDFHIS – 2	NACDFHIS – 1	NACDFEGIS – 2

Final flow 34

d e.g.



e and **f** GI EI and HI

AB CE (EF) HG and HI

Solutionbank D2
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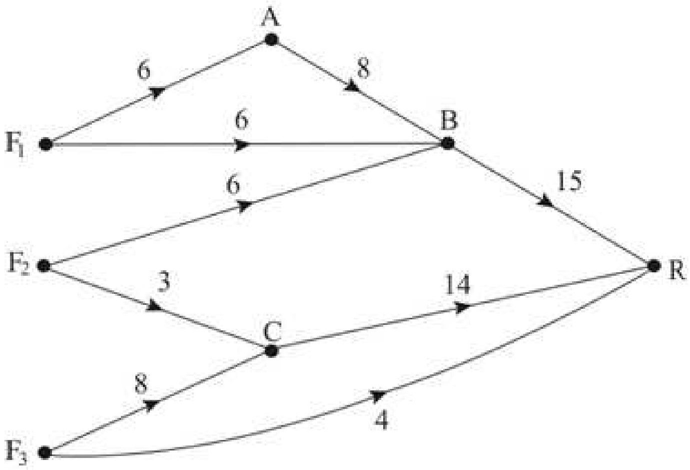
Exercise F, Question 4

Question:

A company wishes to transport its products from 3 factories F_1, F_2 and F_3 to a single retail outlet R. The capacities of the possible routes, in van loads per day, are shown.



a On the worksheet add a supersource S to obtain a capacitated network with a single source and a single sink. State the minimum capacity of each arc you have added.

b **i** State the maximum flow along SF_1ABR and SF_3CR.

ii Show these maximum flows on the worksheet, using numbers in circles.

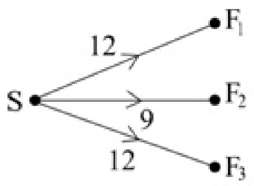
Taking your answer to part **b ii** as the initial flow pattern,

c **i** use the labelling procedure to find a maximum flow from S to R. List each flow-augmenting route you find together with its flow.

ii Prove that your final flow is maximal. ***E***

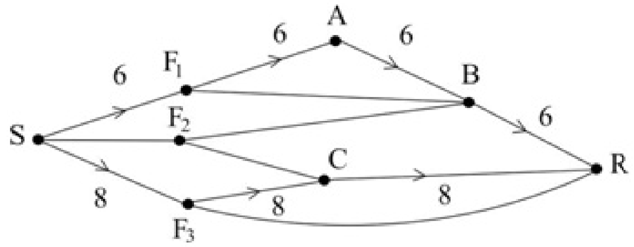
Solution:

a

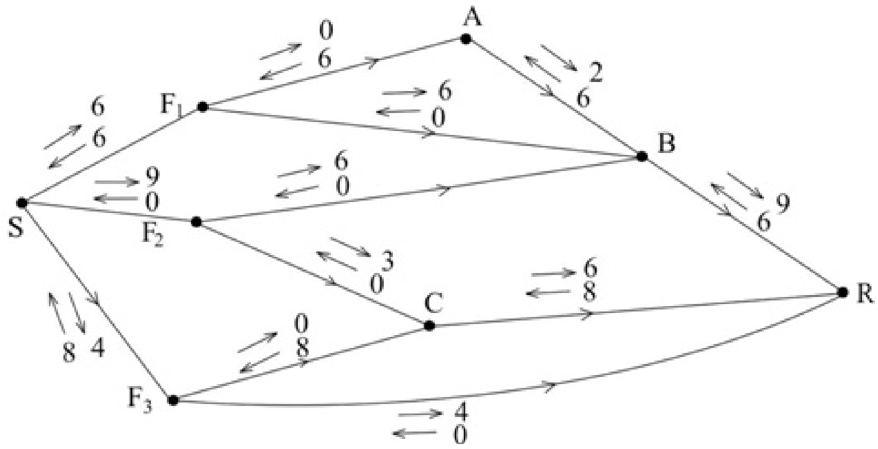


b **i** $SF_1ABR - 6$

ii $SF_3CR - 8$



c **i**



e.g.	$SF_1BR - 6$	or e.g.	$SF_3R - 4$	Total flow 30
	$SF_2BR - 3$		$SF_2BR - 6$	
	$SF_2CR - 3$		$SF_2CR - 3$	
	$SF_3R - 4$		$SF_1BR - 3$	

ii max flow – min cut theorem

e.g. cut BR, F_2C, F_3C, F_3R

(accept BR, F_2C, SF_3)

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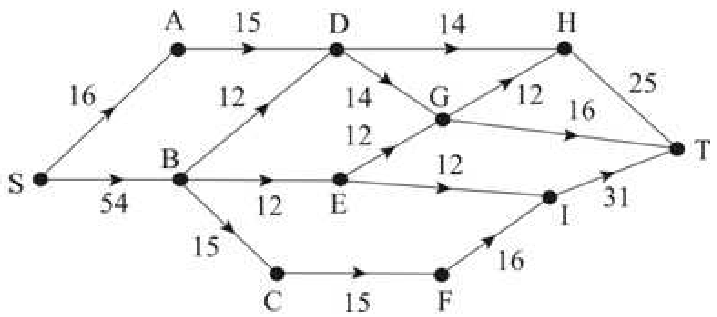
Exercise F, Question 5

Question:

The network represents a road system through a town. The number on each arc represents the maximum number of vehicles that can pass along that road every minute, i.e. the capacity of the road.



a State the maximum flow along
 i S B C F I T,
 ii S A D H T.

b Show these maximum flows on a diagram.

c Taking your answer to part **b** as the initial flow pattern, use the labelling procedure to find a maximum flow from S to T. List each flow-augmenting route you find, together with its flow.

d Indicate a maximum flow.

e Prove that your flow is maximal.

The council has funding to improve one of the roads to increase the flow from S to T. It can choose to increase the flow along one of BE, DH or CF.

f Making your reasoning clear, explain which one of these three roads the council should improve, given that it wishes to maximise the flow through the town.

[*E*]

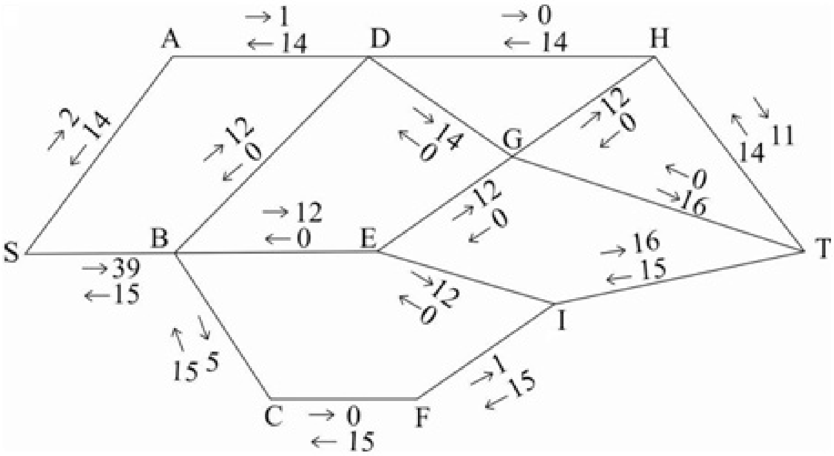
Solution:

a **i** Flow along SBCFIT = 15

ii Flow along SADHT = 14

b A diagram showing the 2 flows correctly

c



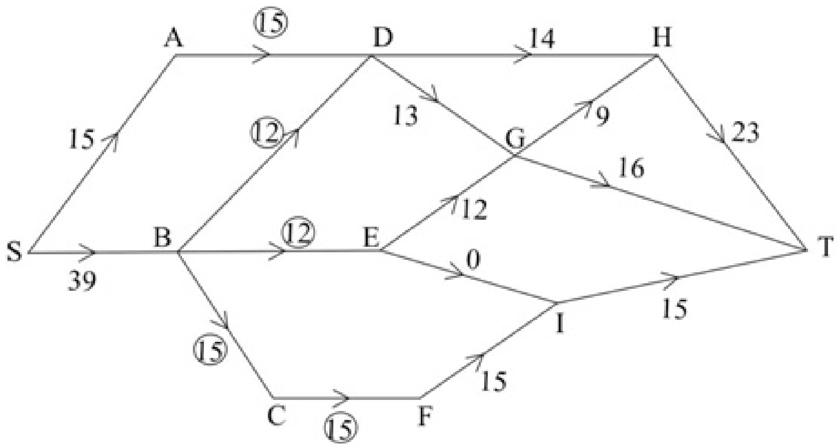
e.g. SADGT − 1

SBDGT − 12

with SBEIT − 12 or SBEGHT − 9 and SBEGT − 3

giving a total flow of 54

d e.g.



e Max flow – min cut theorem, cut through AD, BD, BE and BC or CF

f The flow into D and into C could not increase,
so increase the flow along BE

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Exercise F, Question 6

Question:

Figure 1 shows a capacitated, directed network. The number on each arc indicates the capacity of that arc.

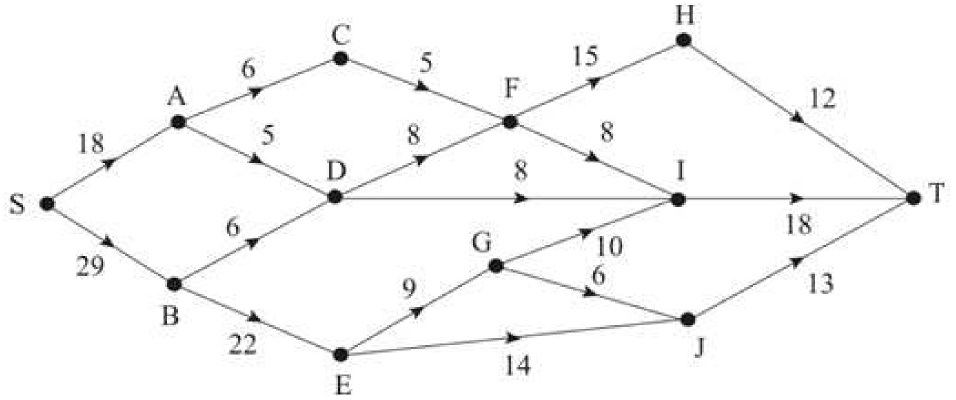


Figure 1

Figure 2 shows a feasible flow of value 29 through the same network.

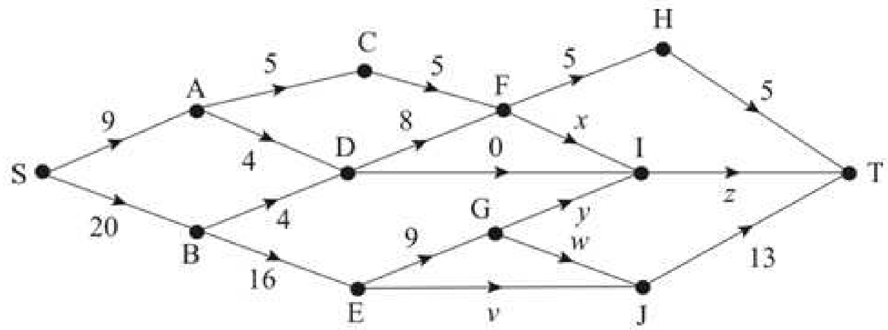


Figure 2

a Find the values of the flows v, w, x, y and z.

Start with the values in Figure 1 and your answers to part **a** as your initial flow pattern.

b Use the labelling procedure on Figure 1 to find the maximum flow through this network, listing each flow–augmenting route you use together with its flow.

c Show the maximum flow on Figure 2 and state its value.

d **i** Find the capacity of the cut which passes through the arcs HT, IT and JT.

ii Find the minimum cut, listing the arcs through which it passes.

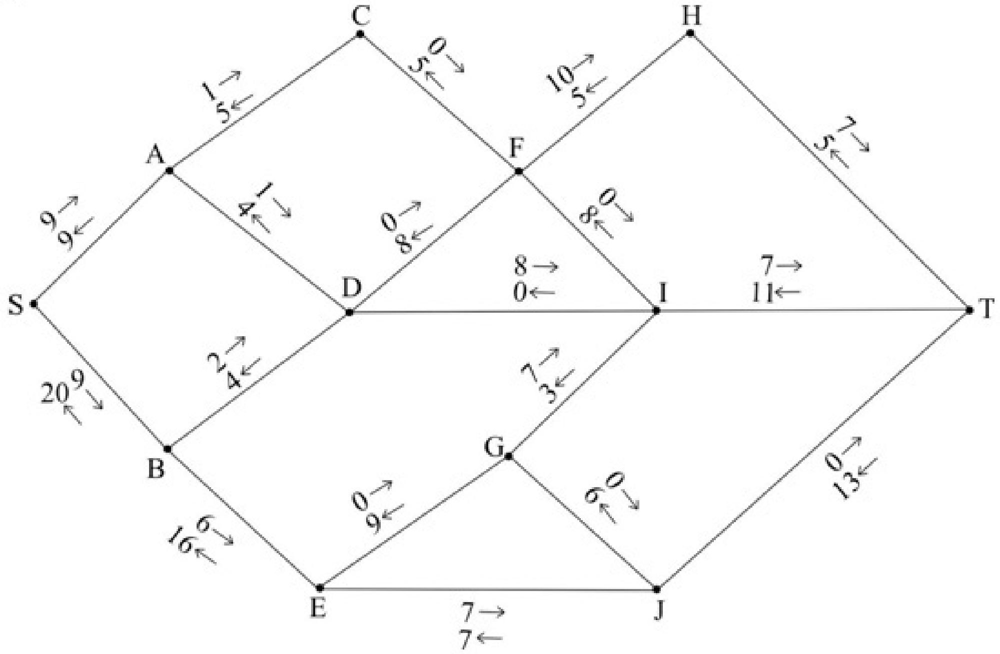
iii Explain why this proves that the flow in part **c** is a maximum.

[*E*]

Solution:

a $v=7$, $w=6$, $x=8$, $y=3$, $z=11$ (conservation of flow)

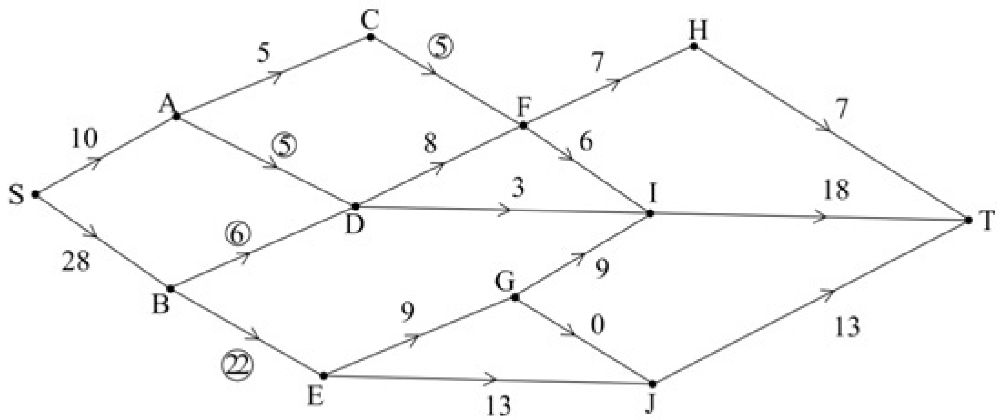
b



Increasing flow by an additional 3
e.g. S B D I T (2)
S A D I T (1)

Additional flow increase (reversing initial flow)
e.g. S B E J G I T (4)
S B E J G I F H T (2)
Flow up to maximum (38)

c E.g.



Complete, consistent flow ringed numbers (flow of 38)

d i $12+18+13=43$

ii CF, AD, BD, BE

iii Max flow – min cut theorem - e.g.

The minimum cut separates the source from the sink. Any additional flow must cross this cut at some point. Since all arcs in the minimum cut are saturated no additional flow can be transported along these arcs. Hence no additional flow is possible.

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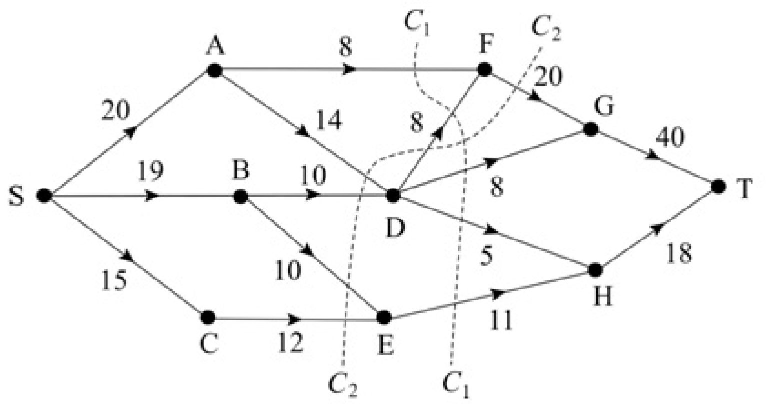
Exercise F, Question 7

Question:



The diagram shows a capacitated, directed network. The number on each arc indicates the capacity of that arc.

a Calculate the values of cuts C_1 and C_2.

Given that one of these cuts is a minimum cut,

b state the maximum flow.

c Deduce the flow along GT, making your reasoning clear.

d By considering the flow into D, deduce that there are only two possible integer values for the flow along SA.

e For each of the two values found in part **d**, draw a complete maximum flow pattern.

f Given that the flow along each arc must be an integer, determine the number of other maximum flow patterns. Give a reason for your answer.

[*E*]

Solution:

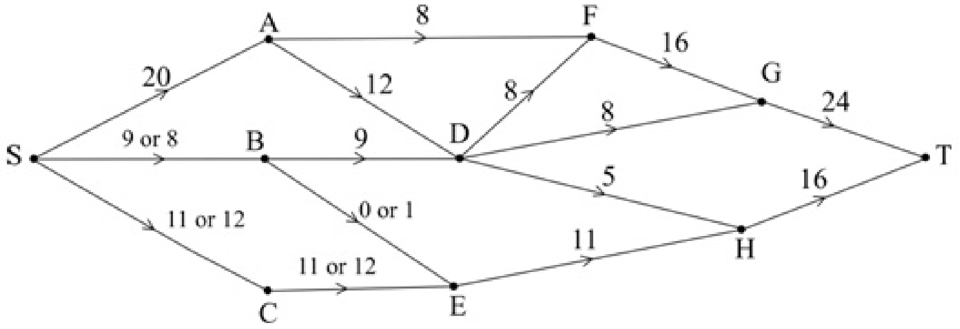
a $C_1 - 40 \quad C_2 - 56$

b max flow = min cut = 40

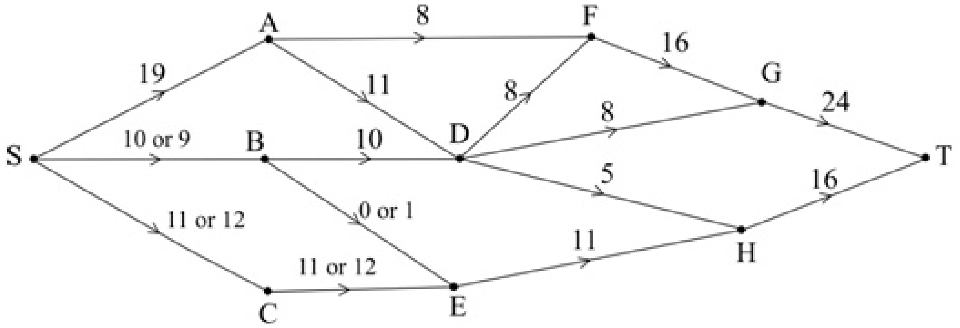
c e.g. Flow into F is 16 $\therefore$ flow into G is 24. The flow along DG is 8 $\therefore$ Flow along GT is 24

d e.g. Flow into A = flow out of A $\therefore$ flow along $AD \le 12$
Flow into D = flow out of D = 21
So flow along AD + flow along BD = 21
$\therefore$ flow along AD and BD could be 12 + 9 or 11 + 10
$\therefore$ possible flows are 20 and 19

e SA = 20



SA = 19



f There are 2 more - CE could be 11 or 12 in each case, for example.

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Exercise A, Question 1

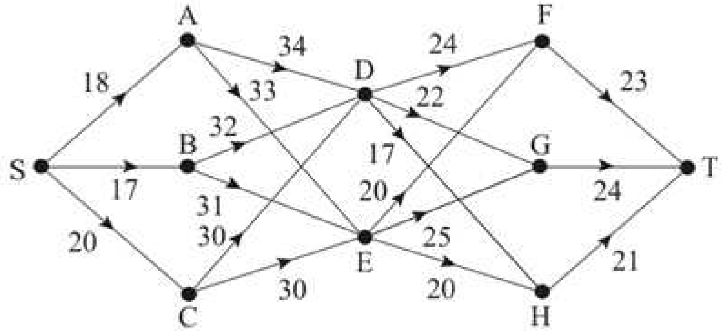
Question:

Use dynamic programming to find

a a shortest

b a longest

route from S to T in the network below. State the route and its length.



Solution:

a Shortest

Stage	State	Action	Destination	Value
1	F	FT	T	23^*
	G	GT	T	24^*
	H	HT	T	21^*
2	D	DF	F	$24+23=47$
		DG	G	$22+24=46$
		DH	H	$17+21=38^*$
	E	EF	F	$20+23=43$
		EG	G	$25+24=49$
		EH	H	$20+21=41^*$
3	A	AD	D	$34+38=72^*$
		AE	E	$33+41=74$
	B	BD	D	$32+38=70^*$
		BE	E	$31+41=72$
	C	CD	D	$30+38=68^*$
		CE	E	$30+41=71$
4	S	SA	A	$18+72=90$
		SB	B	$17+70=87^*$
		SC	C	$20+68=88$

Shortest route SBDHT length 87

b Longest

Stage	State	Action	Destination	Value
1	F	FT	T	23^*
	G	GT	T	24^*
	H	HT	T	21^*
2	D	DF	F	$24+23=47^*$
		DG	G	$22+24=46$
		DH	H	$17+21=38$
	E	EF	F	$20+23=43$
		EG	G	$25+24=49^*$
		EH	H	$20+21=41$
3	A	AD	D	$34+47=81$
		AE	E	$33+49=82^*$
	B	BD	D	$32+47=79$
		BE	E	$31+49=80^*$
	C	CD	D	$30+47=77$
		CE	E	$30+49=79^*$
4	S	SA	A	$18+82=100^*$
		SB	B	$17+80=97$
		SC	C	$20+79=99$

Longest route SAEGT length 100

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise A, Question 2

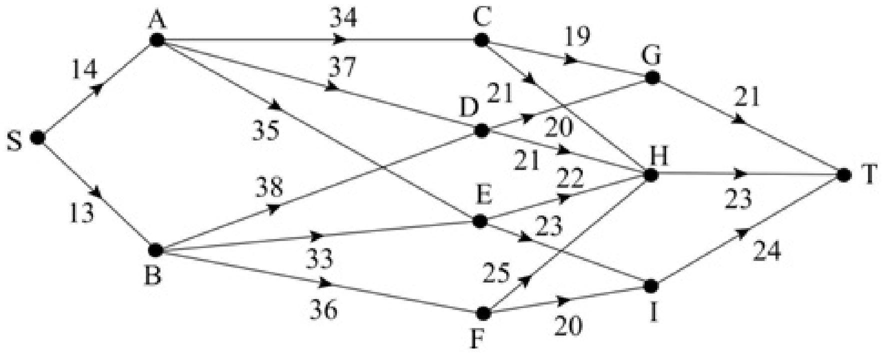
Question:

Use dynamic programming to find

a a shortest

b a longest

route from S to T in the network below. State the route and its length.



Solution:

a Shortest

Stage	State	Action	Destination	Value
1	G	GT	T	21^*
	H	HT	T	23^*
	I	IT	T	24^*
2	C	CG	G	$19+21=40^*$
		CH	H	$21+23=44$
	D	DG	G	$20+21=41^*$
		DH	H	$21+23=44$
	E	EH	H	$22+23=45^*$
		EI	I	$23+24=47$
	F	FH	H	$25+23=48$
		FI	I	$20+24=44^*$
3	A	AC	C	$34+40=74^*$
		AD	D	$37+41=78$
		AE	E	$35+45=80$
	B	BD	D	$38+41=79$
		BE	E	$33+45=78^*$
		BF	F	$36+44=80$
4	S	SA	A	$14+74=88^*$
		SB	B	$13+78=91$

Shortest route length is 88 with route SACGT

b Longest

Stage	State	Action	Destination	Value
1	G	GT	T	21^*
	H	HT	T	23^*
	I	IT	T	24^*
2	C	CG	G	$19+21=40$
		CH	H	$21+23=44^*$
	D	DG	G	$20+21=41$
		DH	H	$21+23=44^*$
	E	EH	H	$22+23=45$
		EI	I	$23+24=47^*$
	F	FH	H	$25+23=48^*$
		FI	I	$20+24=44$
3	A	AC	C	$34+44=78$
		AD	D	$37+44=81$
		AE	E	$35+47=82^*$
	B	BD	D	$38+44=82$
		BE	E	$33+47=80$
		BF	F	$36+48=84^*$
4	S	SA	A	$14+82=96$
		SB	B	$13+84=97^*$

Longest route length is 97 with route SBFHT

Solutionbank D2
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Exercise A, Question 3

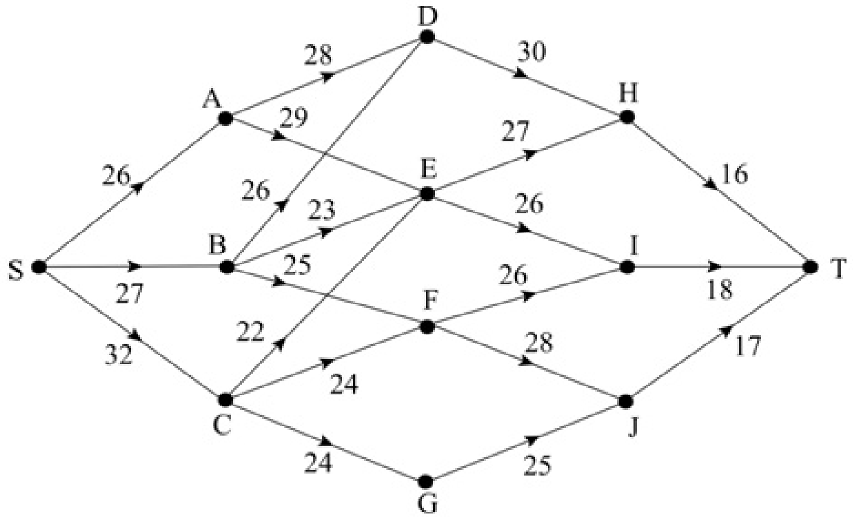
Question:

Use dynamic programming to find

a a shortest

b a longest

route from S to T in the network below. State the route and its length.



Solution:

a Shortest

Stage	State	Action	Destination	Value
1	H	HT	T	16^*
	I	IT	T	18^*
	J	JT	T	17^*
2	D	DH	H	$30+16=46^*$
	E	EH	H	$27+16=43^*$
		EI	I	$26+18=44$
	F	FI	I	$26+18=44^*$
		FJ	J	$28+17=45$
	G	GJ	J	$25+17=42^*$
3	A	AD	D	$28+46=74$
		AE	E	$29+43=72^*$
	B	BD	D	$26+46=72$
		BE	E	$23+43=66^*$
		BF	F	$25+44=69$
	C	CE	E	$22+43=65^*$
		CF	F	$24+44=68$
		CG	G	$24+42=66$
4	S	SA	A	$26+72=98$
		SB	B	$27+66=93^*$
		SC	C	$32+65=97$

Shortest route length is 93 with route SBEHT

b Longest

Stage	State	Action	Destination	Value
1	H	HT	T	16^*
	I	IT	T	18^*
	J	JT	T	17^*
2	D	DH	H	$30+16=46^*$
	E	EH	H	$27+16=43$
		EI	I	$26+18=44^*$
	F	FI	I	$26+18=44$
		FJ	J	$28+17=45^*$
	G	GJ	J	$25+17=42^*$
3	A	AD	D	$28+46=74^*$
		AE	E	$29+44=73$
	B	BD	D	$26+46=72^*$
		BE	E	$23+44=67$
		BF	F	$25+45=70$
	C	CE	E	$22+44=66$
		CF	F	$24+45=69^*$
		CG	G	$24+42=66$
4	S	SA	A	$26+74=100$
		SB	B	$27+72=99$
		SC	C	$32+69=101^*$

Longest route length is 101 with route SCFJT

Solutionbank D2
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Exercise A, Question 4

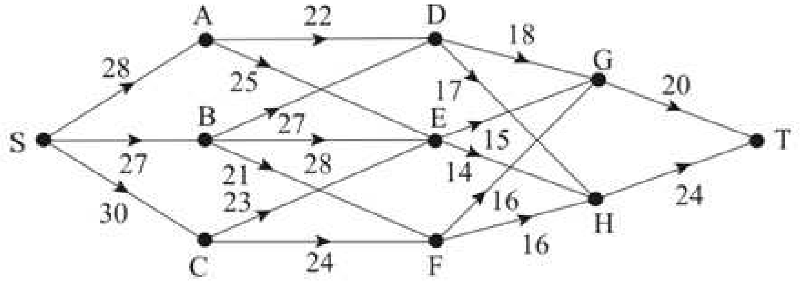
Question:

Use dynamic programming to find

a a shortest

b a longest

route from S to T in the network below. State the route and its length.



Solution:

a Shortest

Stage	State	Action	Destination	Value
1	G	GT	T	20*
	H	HT	T	24*
2	D	DG	G	$18+20=38^*$
		DH	H	$17+24=41$
	E	EG	G	$15+20=35^*$
		EH	H	$14+24=38$
	F	FG	G	$16+20=36^*$
		FH	H	$16+24=40$
3	A	AD	D	$22+38=60^*$
		AE	E	$25+35=60^*$
	B	BD	D	$27+38=65$
		BE	E	$28+35=63$
		BF	F	$21+36=57^*$
	C	CE	E	$23+35=58^*$
		CF	F	$24+36=60$
4	S	SA	A	$28+60=88$
		SB	B	$27+57=84^*$
		SC	C	$30+58=88$

Shortest route length is 84 with route SBFGT

b Longest

Stage	State	Action	Destination	Value
1	G	GT	T	20*
	H	HT	T	24*
2	D	DG	G	$18+20=38$
		DH	H	$17+24=41^*$
	E	EG	G	$15+20=35$
		EH	H	$14+24=38^*$
	F	FG	G	$16+20=36$
		FH	H	$16+24=40^*$
3	A	AD	D	$22+41=63^*$
		AE	E	$25+38=63^*$
	B	BD	D	$27+41=68^*$
		BE	E	$28+38=66$
		BF	F	$21+40=61$
	C	CE	E	$23+38=61$
		CF	F	$24+40=64^*$
4	S	SA	A	$28+63=91$
		SB	B	$27+68=95^*$
		SC	C	$30+64=94$

Longest route length is 95 with route SBDHT

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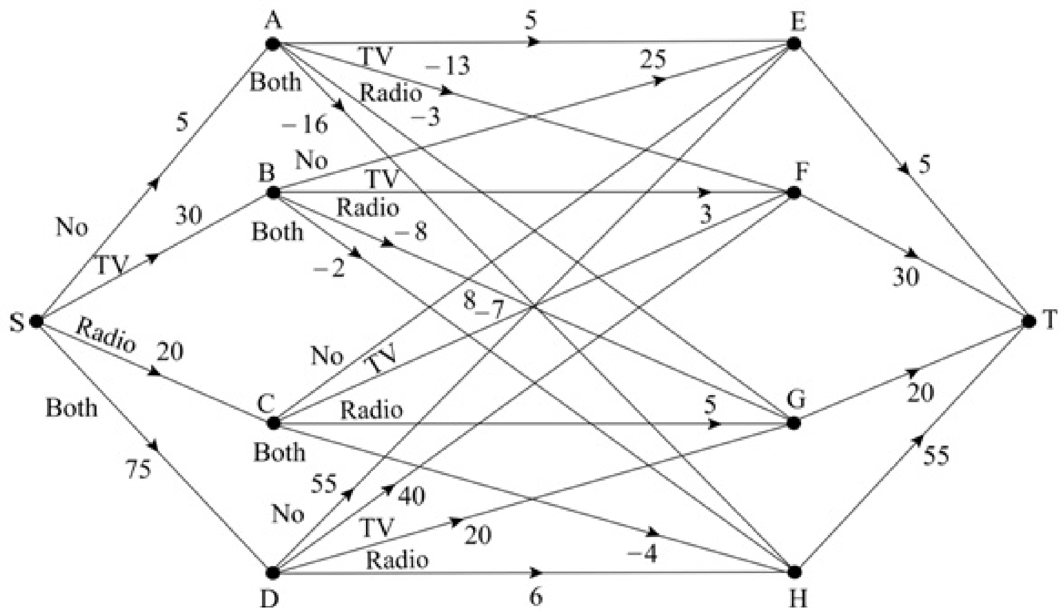
Exercise A, Question 5

Question:



The diagram shows the effect on a company's profits, in £1000's, of taking various advertising decisions. The company wishes to create a two-year plan that will maximise its total profit.

Each year they must decide if they will not advertise (No), advertise through television only (TV), advertise through radio only (Radio), or advertise in both media (Both).

To determine the effectiveness of the strategy the company will estimate the value of its assets at the end of the two-year period.

Use dynamic programming to determine the advertising decisions that the directors should take.

Solution:

Maximise

Stage	State	Action	Destination	Value
1	E	ET	T	5^*
Assets	F	FT	T	30^*
	G	GT	T	20^*
	H	HT	T	55^*
2	A	AE (No)	E	$5+5=10$
Year two		AF (TV)	F	$-13+30=17$
		AG (Radio)	G	$-3+20=17$
		AH (Both)	H	$-16+55=39^*$
	B	BE (No)	E	$25+5=30$
		BF (TV)	F	$3+30=33$
		BG (Radio)	G	$8+20=28$
		BH (Both)	H	$-2+55=53^*$
	C	CE (No)	E	$8+5=13$
		CF (TV)	F	$-7+30=23$
		CG (Radio)	G	$5+20=25$
		CH (Both)	H	$-4+55=51^*$
	D	DE (No)	E	$55+5=60$
		DF (TV)	F	$40+30=70^*$
		DG (Radio)	G	$20+20=40$
		DH (Both)	H	$6+55=61$
3	S	SA (No)	A	$5+39=44$
Year one		SB (TV)	B	$30+53=83$
		SC (Radio)	C	$20+51=71$
		SD (Both)	D	$75+70=145^*$

The maximum profit is £145 000
The maximum route is SDFT
In practical terms the company's strategy is:
Year 1 – advertise in both TV and Radio
Year 2 – advertise on TV only.

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 1

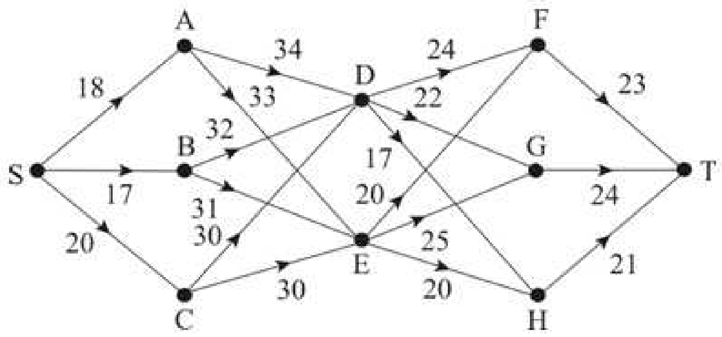
Question:

Use dynamic programming to find

a a minimax,

b a maximin,

route from S to T in the network below. State the route and its length.



Solution:

a Minimax

Stage	State	Action	Destination	Value
1	F	FT	T	23*
	G	GT	T	24*
	H	HT	T	21*
2	D	DF	F	Max(24, 23) = 24
		DG	G	Max(22, 24) = 24
		DH	H	Max(17, 21) = 21*
	E	EF	F	Max(20, 23) = 23
		EG	G	Max(25, 24) = 25
		EH	H	Max(20, 21) = 21*
3	A	AD	D	Max(34, 21) = 34
		AE	E	Max(33, 21) = 33*
	B	BD	D	Max(32, 21) = 32
		BE	E	Max(31, 21) = 31*
	C	CD	D	Max(30, 21) = 30*
		CE	E	Max(30, 21) = 30*
4	S	SA	A	Max(18, 33) = 33
		SB	B	Max(17, 31) = 31
		SC	C	Max(20, 30) = 30*

Minimax route SCDHT or SCEHT – both of value 30

b Maximin

Stage	State	Action	Destination	Value
1	F	FT	T	23^*
	G	GT	T	24^*
	H	HT	T	21^*
2	D	DF	F	$\text{Min}(24, 23) = 23^*$
		DG	G	$\text{Min}(22, 24) = 22$
		DH	H	$\text{Min}(17, 21) = 17$
	E	EF	F	$\text{Min}(20, 23) = 20$
		EG	G	$\text{Min}(25, 24) = 24^*$
		EH	H	$\text{Min}(20, 21) = 20$
3	A	AD	D	$\text{Min}(34, 23) = 23$
		AE	E	$\text{Min}(33, 24) = 24^*$
	B	BD	D	$\text{Min}(32, 23) = 23$
		BE	E	$\text{Min}(31, 24) = 24^*$
	C	CD	D	$\text{Min}(30, 23) = 23$
		CE	E	$\text{Min}(30, 24) = 24^*$
4	S	SA	A	$\text{Min}(18, 24) = 18$
		SB	B	$\text{Min}(17, 24) = 17$
		SC	C	$\text{Min}(20, 24) = 20^*$

Maximin route SCEGT of value 20

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Exercise B, Question 2

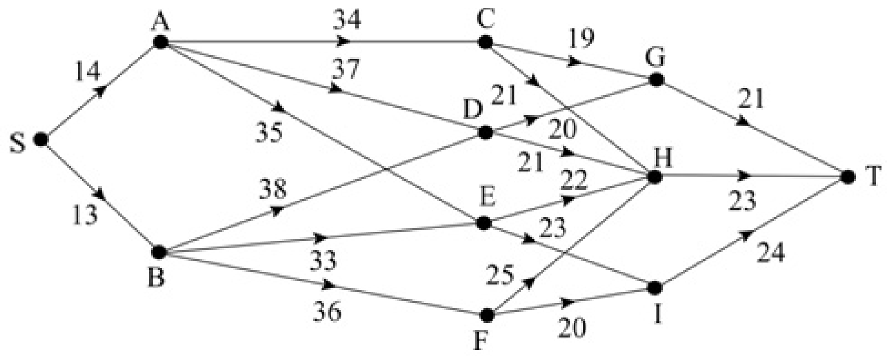
Question:

Use dynamic programming to find

a a minimax,

b a maximin,

route from S to T in the network below. State the route and its length.



Solution:

a Minimax

Stage	State	Action	Destination	Value
1	G	GT	T	21*
	H	HT	T	23*
	I	IT	T	24*
2	C	CG	G	Max(19, 21) = 21*
		CH	H	Max(21, 23) = 23
	D	DG	G	Max(20, 21) = 21*
		DH	H	Max(21, 23) = 23
	E	EH	H	Max(22, 23) = 23*
		EI	I	Max(23, 24) = 24
	F	FH	H	Max(25, 23) = 25
		FI	I	Max(20, 24) = 24*
3	A	AC	C	Max(34, 21) = 34*
		AD	D	Max(37, 21) = 37
		AE	E	Max(35, 23) = 35
	B	BD	D	Max(38, 21) = 38
		BE	E	Max(33, 23) = 33*
		BF	F	Max(36, 24) = 36
4	S	SA	A	Max (14, 34) = 34
		SB	B	Max (13, 33) = 33*

Minimax route SBEHT of value 33

b Maximin

Stage	State	Action	Destination	Value
1	G	GT	T	21^*
	H	HT	T	23^*
	I	IT	T	24^*
2	C	CG	G	$\text{Min}(19, 21) = 19$
		CH	H	$\text{Min}(21, 23) = 21^*$
	D	DG	G	$\text{Min}(20, 21) = 20$
		DH	H	$\text{Min}(21, 23) = 21^*$
	E	EH	H	$\text{Min}(22, 23) = 22$
		EI	I	$\text{Min}(23, 24) = 23^*$
	F	FH	H	$\text{Min}(25, 23) = 23^*$
		FI	I	$\text{Min}(20, 24) = 20$
3	A	AC	C	$\text{Min}(34, 21) = 34$
		AD	D	$\text{Min}(37, 21) = 37^*$
		AE	E	$\text{Min}(35, 23) = 35$
	B	BD	D	$\text{Min}(38, 21) = 38^*$
		BE	E	$\text{Min}(33, 23) = 33$
		BF	F	$\text{Min}(36, 24) = 36$
4	S	SA	A	$\text{Min}(14, 37) = 37$
		SB	B	$\text{Min}(13, 38) = 38^*$

Maximin route SBDHT of value 38

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise B, Question 3

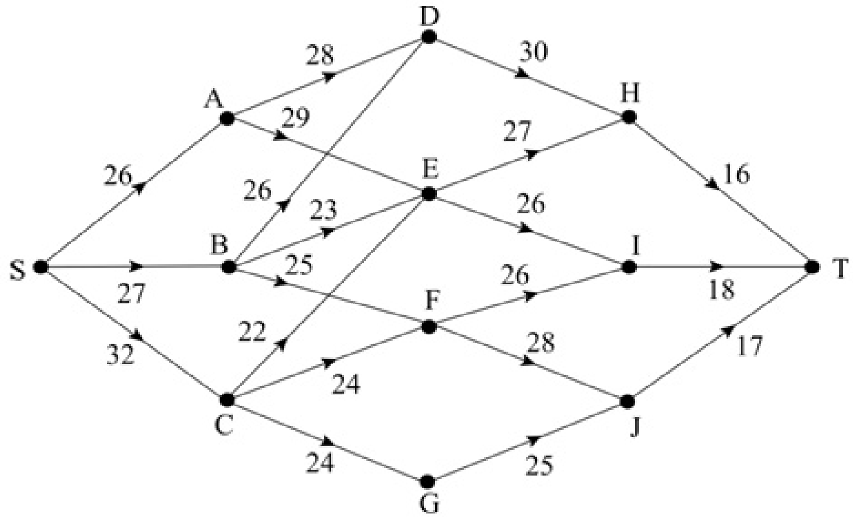
Question:

Use dynamic programming to find

a a minimax,

b a maximin,

route from S to T in the network below. State the route and its length.



Solution:

a Minimax

Stage	State	Action	Destination	Value
1	H	HT	T	16*
	I	IT	T	18*
	J	JT	T	17*
2	D	DH	H	Max(30,16) = 30*
	E	EH	H	Max(27,16) = 27
		EI	I	Max(26,18) = 26*
	F	FI	I	Max(26,18) = 26*
		FJ	J	Max(28,17) = 28
	G	GJ	J	Max(25,17) = 25*
3	A	AD	D	Max(28,30) = 30
		AE	E	Max(29,26) = 29*
	B	BD	D	Max(26,30) = 30
		BE	E	Max(23,26) = 26*
		BF	F	Max(25,26) = 26*
	C	CE	E	Max(22,26) = 26
		CF	F	Max(24,26) = 26
		CG	G	Max(24,25) = 25*
4	S	SA	A	Max(26,29) = 29
		SB	B	Max(27,26) = 27*
		SC	C	Max(32,25) = 32

Minimax route SBEIT or SBFIT – both of value 27

b Maximin

Stage	State	Action	Destination	Value
1	H	HT	T	16*
	I	IT	T	18*
	J	JT	T	17*
2	D	DH	H	Min(30,16) = 16*
	E	EH	H	Min(27,16) = 16
		EI	I	Min(26,18) = 18*
	F	FI	I	Min(26,18) = 18*
		FJ	J	Min(28,17) = 17
	G	GJ	J	Min(25,17) = 17*
3	A	AD	D	Min(28,16) = 16
		AE	E	Min(29,18) = 18*
	B	BD	D	Min(26,16) = 16
		BE	E	Min(23,18) = 18*
		BF	F	Min(25,18) = 18*
	C	CE	E	Min(22,18) = 18*
		CF	F	Min(24,18) = 18*
		CG	G	Min(24,17) = 17
4	S	SA	A	Min(26,18) = 18*
		SB	B	Min(27,18) = 18*
		SC	C	Min(32,18) = 18*

Maximin routes SAEIT, SBEIT, SBFIT, SCEIT, SCFIT all of value 18

Solutionbank D2

Edexcel AS and A Level Modular Mathematics

Exercise B, Question 4

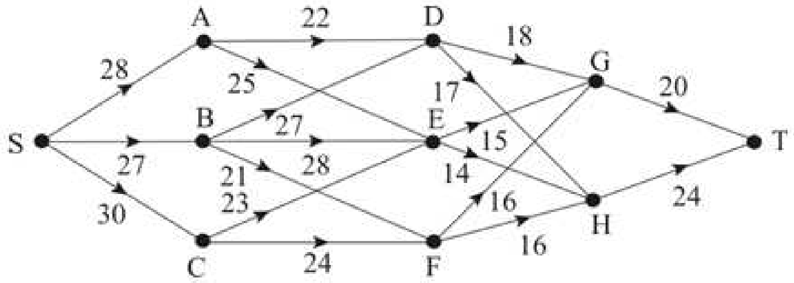
Question:

Use dynamic programming to find

a a minimax,

b a maximin,

route from S to T in the network below. State the route and its length.



Solution:

a Minimax

Stage	State	Action	Destination	Value
1	G	GT	T	20*
	H	HT	T	24*
2	D	DG	G	Max(18, 20) = 20*
		DH	H	Max(17, 24) = 24
	E	EG	G	Max(15, 20) = 20*
		EH	H	Max(14, 24) = 24
	F	FG	G	Max(16, 20) = 20*
		FH	H	Max(16, 24) = 24
3	A	AD	D	Max(22, 20) = 22*
		AE	E	Max(25, 20) = 25
	B	BD	D	Max(27, 20) = 27
		BE	E	Max(28, 20) = 28
		BF	F	Max(21, 20) = 21*
	C	CE	E	Max(23, 20) = 23*
		CF	F	Max(24, 20) = 24
4	S	SA	A	Max(28, 22) = 28
		SB	B	Max(27, 21) = 27*
		SC	C	Max(30, 23) = 30

Minimax route SBFGT of value 27

b Maximin

Stage	State	Action	Destination	Value
1	G	GT	T	20*
	H	HT	T	24*
2	D	DG	G	$\text{Min}(18,20)=18^*$
		DH	H	$\text{Min}(17,24)=17$
	E	EG	G	$\text{Min}(15,20)=15^*$
		EH	H	$\text{Min}(14,24)=14$
	F	FG	G	$\text{Min}(16,20)=16^*$
		FH	H	$\text{Min}(16,24)=16^*$
3	A	AD	D	$\text{Min}(22,18)=18^*$
		AE	E	$\text{Min}(25,15)=15$
	B	BD	D	$\text{Min}(27,18)=18^*$
		BE	E	$\text{Min}(28,15)=15$
		BF	F	$\text{Min}(21,16)=16$
	C	CE	E	$\text{Min}(23,15)=15$
		CF	F	$\text{Min}(24,16)=16^*$
4	S	SA	A	$\text{Min}(28,18)=18^*$
		SB	B	$\text{Min}(27,18)=18^*$
		SC	C	$\text{Min}(30,16)=16$

Maximin routes SADGT and SBDGT both of value 18

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 1

Question:

A company is created to sell holidays on an island. There are three new resorts A, B and C being created on the island and the company decides to introduce one new resort to its catalogue each year over the next three years. The costs of introducing each resort will be influenced by the number of resorts listed in the catalogue. The more resorts the company has listed, the smaller the cost of adding another resort. The estimates of annual costs are shown in the table below, in hundreds of pounds.

Resorts listed	A	B	C
None	60	60	55
A	–	50	60
B	40	–	55
C	35	50	–
A and B	–	–	50
A and C	–	45	–
B and C	30	–	–

For funding reasons the company needs to choose the order in which the resorts are introduced so that the greatest annual cost is as small as possible.
Dynamic programming will be used to determine the order in which the resorts are introduced.

a Explain the meaning of Stage, State and Action in this context.

b Find the order in which the resorts should be added and the greatest annual cost.

Solution:

a Stage – time, in years, remaining
State – resorts already created
Action – resort to be opened

b We require the route that gives the minimax value

Stage	State	Action	Destination	Value
1	AB	C	ABC	50*
	AC	B	ABC	45*
	BC	A	ABC	30*
2	A	B	AB	$\text{Max}(50,50) = 50^*$
		C	AC	$\text{Max}(60,45) = 60$
	B	A	AB	$\text{Max}(40,50) = 50^*$
		C	BC	$\text{Max}(55,30) = 55$
	C	A	AC	$\text{Max}(35,45) = 45^*$
		B	BC	$\text{Max}(50,30) = 50$
3	none	A	A	$\text{Max}(60,50) = 60$
		B	B	$\text{Max}(60,50) = 60$
		C	C	$\text{Max}(55,45) = 55^*$

The minimax route is CAB with a value of £5500
The order in which the results should be built is C then A then B

Solutionbank D2
Edexcel AS and A Level Modular Mathematics

Exercise C, Question 2

Question:

A house renovation project is to be completed in 6 weeks (30 working days). The work is in four phases: clearance, repairing, modernisation and decorating, which must be undertaken in that order. The cost, in £1000, of each stage depends on the time taken to do it. These are shown in the table.

Time for stage (days)	Clearance	Repairing	Modernisation	Decorating
5	15	24	22	14
10	13	20	19	12
15	8	15	15	9
20	5	10	11	4
25	2	6	7	2

Dynamic programming will be used to solve this problem.

a Define the terms Stage, State, Action, Destination and Value in this context.

b Determine the number of days that should be allocated to each stage in order to minimise costs.

Solution:

a Stage – phase being considered
State – number of days remaining
Action – number of days allocated
Destination – number of days remaining
Value – total costs

b

Stage	State	Action	Destination	Value
Decorating	5	5	0	14^*
	10	10	0	12^*
Modernisation	10	5	5	$22+14=36^*$
	15	10	5	$19+14=33^*$
		5	10	$22+12=34$
Repairing	15	5	10	$24+36=60^*$
	20	10	10	$20+36=56^*$
		5	15	$24+33=57$
Clearance	25	10	15	$13+60=73$
		5	20	$15+56=71^*$

The minimum cost is £71 000. The time should be allocated as follows

Activity	Clearance	Repairing	Modernisation	Decorating
Number of days	5	10	5	5

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Exercise C, Question 3

Question:

A company makes aircraft. The order book over the next four months is shown in the table below.

Month	March	April	May	June
Number of aircraft ordered	1	2	3	2

The aircraft are delivered to customers at the end of each month.

Up to three aircraft can be made in any month, but if more than two are made in any one month additional equipment will need to be hired at £20 000 per month.

If any work is done in a month the overhead costs are £50 000.

Up to two aircraft can be held in secure hangers at a cost of £10 000 per aircraft per month.

There are no aircraft in store at the beginning of March and there should be no aircraft in store after the June delivery.

Use dynamic programming to determine the production schedule that minimises the costs.

Solution:

Stage – Month
State – number in storage
Action – number to be made

Stage	State	Action	Destination	Value (in £10000)
June	2	0	0	$2 = 2^*$
(2)	1	1	0	$5+1 = 6^*$
	0	2	0	$5 = 5^*$
May	2	1	0	$5+2+5 = 12$
(3)		2	1	$5+2+6 = 13$
		3	2	$2+5+2+2 = 11^*$
	1	2	0	$5+1+5 = 11^*$
		3	1	$2+5+1+6 = 14$
	0	3	0	$2+5+1+2 = 10^*$
April	2	0	0	$2+10 = 12^*$
(2)		1	1	$5+2+11 = 18$
		2	2	$5+2+11 = 18$
	1	1	0	$5+1+10 = 16^*$
		2	1	$5+1+11 = 17$
		3	2	$2+5+1+11 = 19$
	0	2	0	$5+10 = 15^*$
		3	1	$2+5+11 = 18$
March	0	1	0	$5+15 = 20$
(1)		2	1	$5+16 = 21$
		3	2	$2+5+12 = 19^*$

The minimum cost is £190 000, the aircraft should be built as follows.

Month	March	April	May	June
Number of aircraft built in each month	3	0	3	2

Solutionbank D2

Edexcel AS and A Level Modular Mathematics

Exercise C, Question 4

Question:

A salesman will visit four shops in the next four days to demonstrate a new product. He will start at home and travel to the first shop and spend the first day there, then travel directly to the second shop for day 2, onto the third shop for day 3, then to the fourth shop for day 4 and then travel home.

Table 1 shows the shops he could visit on each day.
Table 2 shows the anticipated profit, in £100, from sales at each shop.
Table 3 shows the travelling expenses, in £100, that will be incurred.

The company employing the salesman wishes to maximise the income, after subtracting the travel costs, generated by the salesman's visits. Find his optimum route.

Table 1

Monday	Tuesday	Wednesday	Thursday
A, B, C	D, E	F, G	H, I, J

Table 2

Shop	A	B	C	D	E	F	G	H	I	J
Profit	8	9	8	12	14	10	11	14	13	11

Table 3

	A	B	C	D	E	F	G	H	I	J
Home	2	2	3					6	4	3
A				3	4					
B				4	6					
C				4	4					
D						5	5			
E						4	7			
F								5	4	4
G								5	5	4

Solution:

Stage – day
State – shop being visited
Action – next journey to be undertaken

Stage	State	Action	Destination	Value, in £ 100
Thursday	H	H-home	home	$14-6=8^*$
	I	I–home	home	$13-4=9^*$
	J	J – home	home	$11-3=8^*$
Wednesday	F	FH	H	$10-5+8=13$
		FI	I	$10-4+9=15^*$
		FJ	J	$10-4+8=14$
	G	GH	H	$11-5+8=14$
		GI	I	$11-5+9=15^*$
		GJ	J	$11-4+8=15^*$
Tuesday	D	DF	F	$12-5+15=22^*$
		DG	G	$12-5+15=22^*$
	E	EF	F	$14-4+15=25^*$
		EG	G	$14-7+15=22$
Monday	A	AD	D	$8-3+22=27$
		AE	E	$8-4+25=29^*$
	B	BD	D	$9-4+22=27$
		BE	E	$9-6+25=28^*$
	C	CD	D	$8-4+22=26$
		CE	E	$8-4+25=29^*$
Sunday	Home	Home – A		$-2+29=27$
		Home – B		$-2+28=26^*$
		Home – C		$-3+29=26^*$

There are two possible minimum routes.
Home – B – E – F – I – Home
Home – C – E – F – I – Home
Each has a value of £2600

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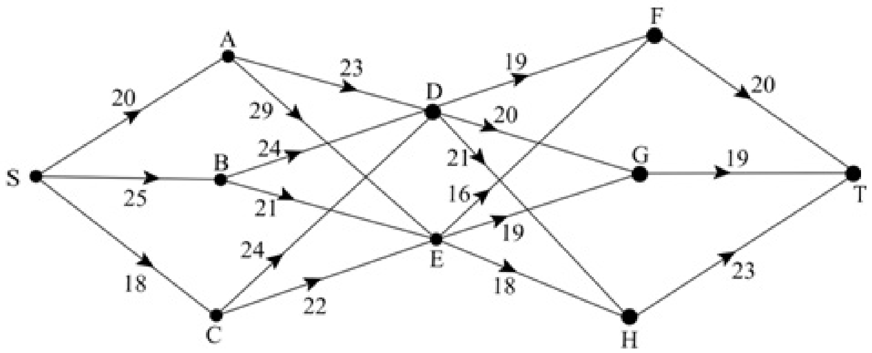
Exercise D, Question 1

Question:



Use dynamic programming to find the minimax route from S to T in the network above.

Solution:

Minimax

Stage	State	Action	Destination	Value
1	F	FT	T	20*
	G	GT	T	19*
	H	HT	T	23*
2	D	DF	F	Max(19, 20) = 20*
		DG	G	Max(20,19) = 20*
		DH	H	Max(21, 23) = 23
	E	EF	F	Max(16, 20) = 20
		EG	G	Max(19,19) = 19*
		EH	H	Max(18, 23) = 23
	A	AD	D	Max(23, 20) = 23*
		AE	E	Max(29,19) = 29
	B	BD	D	Max(24, 20) = 24
		BE	E	Max(21,19) = 21*
	C	CD	D	Max(24, 20) = 24
		CE	E	Max(22,19) = 22*
3	S	SA	A	Max(20, 23) = 23
		SB	B	Max(25, 21) = 25
		SC	C	Max(18, 22) = 22*

Minimax route is SCEGT value 22

Solutionbank D2

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Exercise D, Question 2

Question:

Using the same diagram as in question 1, use dynamic programming to find the maximin route.

Solution:

Maximin

Stage	State	Action	Destination	Value
1	F	FT	T	20*
	G	GT	T	19*
	H	HT	T	23*
2	D	DF	F	$\text{Min}(19, 20) = 19$
		DG	G	$\text{Min}(20, 19) = 19$
		DH	H	$\text{Min}(21, 23) = 21^*$
	E	EF	F	$\text{Min}(16, 20) = 16$
		EG	G	$\text{Min}(19, 19) = 19^*$
		EH	H	$\text{Min}(18, 23) = 18$
	A	AD	D	$\text{Min}(23, 21) = 21^*$
		AE	E	$\text{Min}(29, 19) = 19$
	B	BD	D	$\text{Min}(24, 21) = 21^*$
		BE	E	$\text{Min}(21, 19) = 19$
	C	CD	D	$\text{Min}(24, 21) = 21^*$
		CE	E	$\text{Min}(22, 19) = 19$
3	S	SA	A	$\text{Min}(20, 21) = 20$
		SB	B	$\text{Min}(25, 21) = 21^*$
		SC	C	$\text{Min}(18, 21) = 18$

The maximin route is SBDHT of value 21

Solutionbank D2
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Exercise D, Question 3

Question:

A dairy manufacturer can make butter, cheese and yoghurt. Up to five units of milk can be processed and the profits from the various allocations are shown in the table.

Number of units	1	2	3	4	5
Butter	14	25	34	41	47
Cheese	12	30	40	45	49
Yoghurt	10	20	30	40	50

The manufacturer wishes to maximise his profit.

a Use dynamic programming to find an optimal solution and state the profit.

b Show that there is a second optimal solution.

Solution:

Maximum

Stage	State	Action	Destination	Value
Yoghurt	5	5	0	50^*
	4	4	0	40^*
	3	3	0	30^*
	2	2	0	20^*
	1	1	0	10^*
	0	0	0	0^*
Cheese	5	0	5	$0+50=50$
		1	4	$12+40=52$
		2	3	$30+30=60^*$
		3	2	$40+20=60^*$
		4	1	$45+10=55$
		5	0	$49+0=49$
	4	0	4	$0+40=40$
		1	3	$12+30=42$
		2	2	$30+20=50^*$
		3	1	$40+10=50^*$
		4	0	$45+0=45$
	3	0	3	$0+30=30$
		1	2	$12+20=32$
		2	1	$30+10=40^*$
		3	0	$40+0=40^*$
	2	0	2	$0+20=20$
		1	1	$12+10=22$
		2	0	$30+0=30^*$
	1	0	1	$0+10=10$
		1	0	$12+0=12^*$
	0	0	0	$0+0=0^*$
Butter	5	0	5	$0+60=60$
		1	4	$14+50=64$
		2	3	$25+40=65^*$
		3	2	$34+30=64$
		4	1	$41+12=53$
		5	0	$47+0=47$

There are two possible courses of action each of value £65

Product	Butter	Cheese	Yoghurt
Units to be used	2	2	1
Product	Butter	Cheese	Yoghurt
Units to be used	2	3	0

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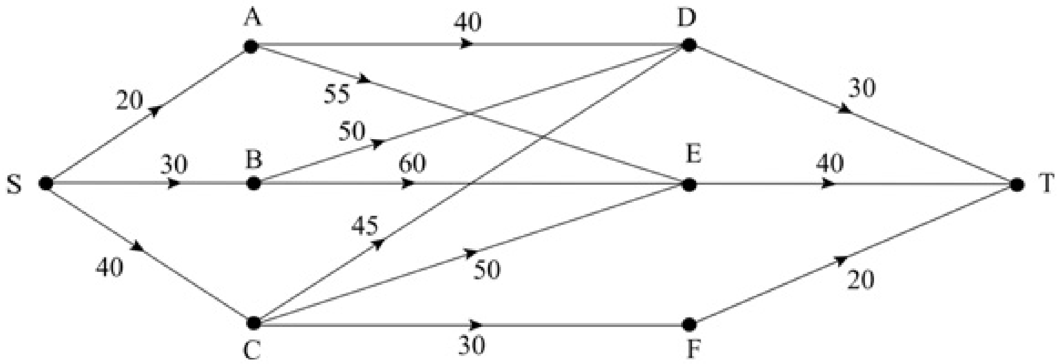
Exercise D, Question 4

Question:

Jenny wishes to travel from S to T. There are several routes available. She wishes to choose the route on which the maximum altitude, above sea level, is as small as possible. This is called the minimax route.



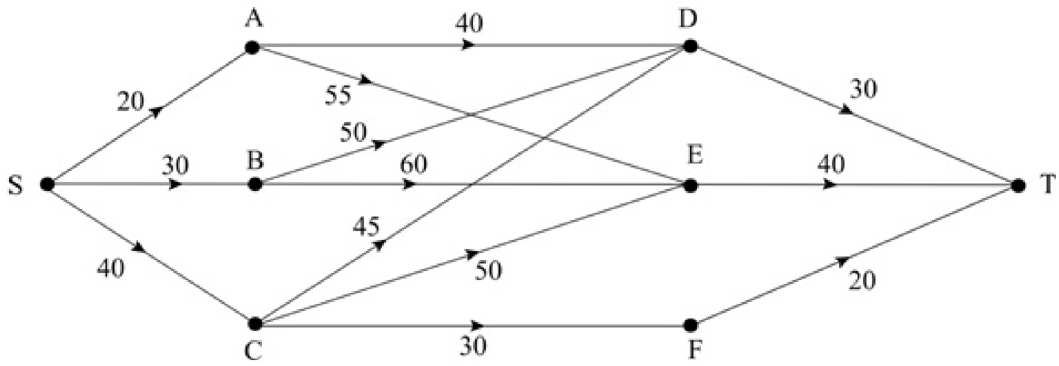
The diagram gives the possible routes and the weights on the edges give the maximum altitude on the road (in units of 100 feet).

Use dynamic programming, carefully defining the stages and states, to determine the route or routes Jenny should take. You should show your calculations in tabular form, using a table with columns labelled as shown below.

Stage	Initial state	Action	Final state	Value

[*E*]

Solution:



The states are the vertices.

Stage	Initial state	Action	Final state	Value
1	D	DT	T	30
	E	ET	T	40
	F	FT	T	20
2	A	AD	D	max (40, 30) = 40*
		AE	E	max (55, 40) = 55
	B	BD	D	max (50, 30) = 50*
		BE	E	max (60, 40) = 60
	C	CD	D	max (45, 30) = 45
		CE	E	max (50, 40) = 50
		CF	F	max (30, 20) = 30*
3	S	SA	A	max (40, 20) = 40*
		SB	B	max (50, 30) = 50
		SC	C	max (40, 30) = 40*

Tracing back there are two routes
SC, CF, FT, $\Rightarrow$ SCFT
SA, AD, DT, $\Rightarrow$ SADT
Maximum altitude on these routes is 40 $(\times 100 \text{ ft}) = 4000$ ft.

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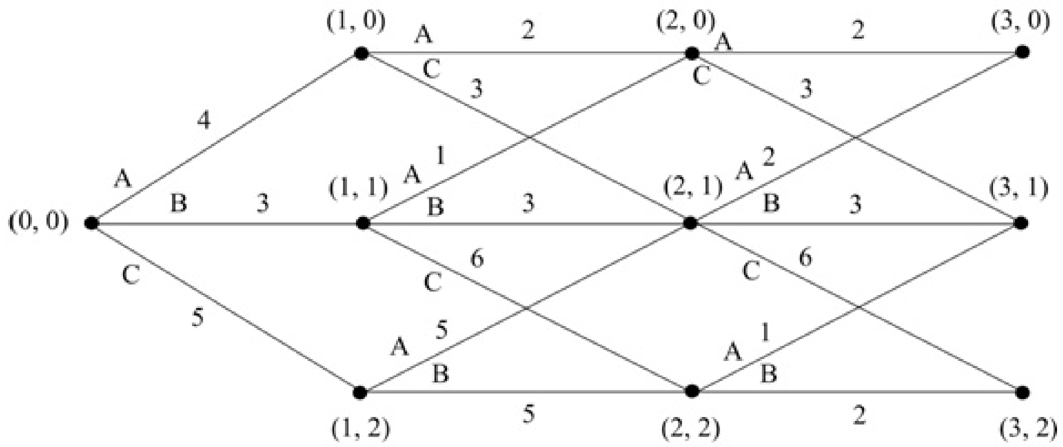
Exercise D, Question 5

Question:

At the beginning of each month an advertising manager must choose one of 3 adverts:
the previous advert;
the current advert;
a new advert.

She therefore has 3 options:
A: use the previous advert;
B: use the current advert;
C: run a new advert.

The possible choices are shown in the network below together with (stage, state) variables at the vertices and the expected profits, in thousands of pounds, on the arcs.



September	October	November

The manager wants to maximise her profits for the three-month period.

a Complete the table on the worksheet.

b Hence obtain the sequence of decisions she should make to obtain the maximum profit. State the maximum profit.

[*E*]

Solution:

a

Stage	State	Action	Cost	Total Cost
2	0	A C	2 3	2 3*
	1	A B C	2 3 6	2 3 6*
	2	A B	1 2	1 2*
1	0	A C	2 3	$2 + 3 = 5$ $3 + 6 = 9$*
	1	A B C	1 3 6	$1 + 3 = 4$ $3 + 6 = 9$* $6 + 2 = 8$
	2	A B	5 5	$5 + 6 = 11$* $5 + 2 = 7$
0	0	A B C	4 3 5	$4 + 9 = 13$ $3 + 9 = 12$ $5 + 11 = 16$

b Hence maximum profit is 16
Tracing back through calculations the optimal strategy is CAC

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Review Exercise 1
Exercise A, Question 1

Question:

A theme park has four sites, A, B, C and D, on which to put kiosks. Each kiosk will sell a different type of refreshment. The income from each kiosk depends upon what it sells and where it is located. The table below shows the expected daily income, in pounds, from each kiosk at each site.

	Hot dogs and beef burgers (H)	**Ice cream (I)**	**Popcorn, candyfloss and drinks (P)**	**Snacks and hot drinks (S)**
Site A	267	272	276	261
Site B	264	271	278	263
Site C	267	273	275	263
Site D	261	269	274	257

Reducing rows first, use the Hungarian algorithm to determine a site for each kiosk in order to maximise the total income. State the site for each kiosk and the total expected income. You must make your method clear and show the table after each stage. ***E***

Solution:

To maximise, subtract all entries from $n \geq 278$

e.g. $\begin{bmatrix} 11 & 6 & 2 & 17 \\ 14 & 7 & 0 & 15 \\ 11 & 5 & 3 & 15 \\ 17 & 9 & 4 & 21 \end{bmatrix}$

Reduce rows $\begin{bmatrix} 9 & 4 & 0 & 15 \\ 14 & 7 & 0 & 15 \\ 8 & 2 & 0 & 12 \\ 13 & 5 & 0 & 17 \end{bmatrix}$ then columns $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 6 & 5 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 5 & 3 & 0 & 5 \end{bmatrix}$

Minimum element is 1

$\begin{bmatrix} 0 & 1 & 0 & 2 \\ 5 & 4 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 4 & 2 & 0 & 4 \end{bmatrix}$

Minimum element is 1

$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 5 & 3 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 4 & 1 & 0 & 3 \end{bmatrix}$

Then Minimum element is 1

$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 4 & 2 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$

optimal

$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$

Minimum element is 2
optimal

So $\begin{matrix} A-H \\ B-P \\ C-S \\ D-I \end{matrix}$ or $\begin{matrix} H \\ S \\ I \\ P \end{matrix}$

(both £1077)

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Review Exercise 1
Exercise A, Question 2

Question:

A coach company has 20 coaches. At the end of a given week, 8 coaches are at depot A, 5 coaches are at depot B and 7 coaches are at depot C. At the beginning of the next week, 4 of these coaches are required at depot D, 10 of them at depot E and 6 of them at depot F. The following table shows the distances, in miles, between the relevant depots.

	D	E	F
A	40	70	25
B	20	40	10
C	35	85	15

The company needs to move the coaches between depots at the weekend. The total mileage covered is to be a minimum.
Formulate this information as a linear programming problem.

a State clearly your decision variables.
b Write down the objective function in terms of your decision variables.
c Write down the constraints, explaining what each constraint represents. ***E***

Solution:

a x_{11} number of coaches from A to D
x_{12} number of coaches from A to E
x_{13} number of coaches from A to F
x_{21} number of coaches from B to D
x_{22} number of coaches from B to E
x_{23} number of coaches from B to F
x_{31} number of coaches from C to D
x_{32} number of coaches from C to E
x_{33} number of coaches from C to F

b Minimise $z = 40x_{11} + 70x_{12} + 25x_{13}$
$+20x_{21} + 40x_{22} + 10x_{23}$
$+35x_{31} + 85x_{32} + 15x_{33}$

c Depot A $x_{11} + x_{12} + x_{13} = 8$ (number of coaches at A)
Depot B $x_{21} + x_{22} + x_{23} = 5$ (number of coaches at B)
Depot C $x_{31} + x_{32} + x_{33} = 7$ (number of coaches at C)
Depot D $x_{11} + x_{21} + x_{31} = 4$ (number required at D)
Depot E $x_{12} + x_{22} + x_{32} = 10$ (number required at E)
Depot F $x_{13} + x_{32} + x_{33} = 6$ (number required at F)
Reference to number of coaches at A, B and C = number of coaches at D, E and F

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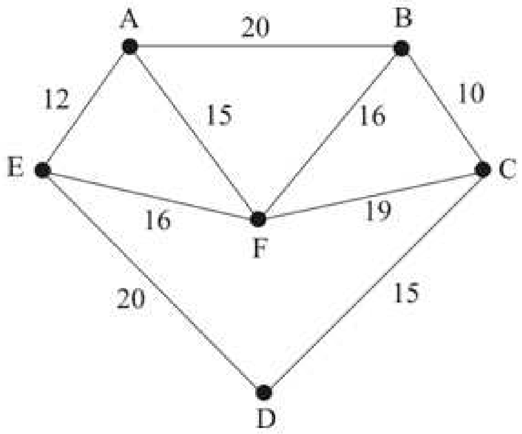
Review Exercise 1
Exercise A, Question 3

Question:



The diagram shows a network of roads connecting six villages A, B, C, D, E and F. The lengths of the roads are given in km.

a Complete the table on the worksheet, in which the entries are the shortest distances between pairs of villages. You should do this by inspection.

The table can now be taken to represent a complete network.

b Use the nearest-neighbour algorithm, starting at A, on your completed table in part **a**. Obtain an upper bound to the length of a tour in this complete network, which starts and finishes at A and visits every village exactly once.

c Interpret your answer in part **b** in terms of the original network of roads connecting the six villages.

d By choosing a different vertex as your starting point, use the nearest-neighbour algorithm to obtain a shorter tour than that found in part **b**.
State the tour and its length. ***E***

Solution:

a

	A	B	C	D	E	F
A	0	20	30	32	12	15
B	20	0	10	25	32	16
C	30	10	0	15	35	19
D	32	25	15	0	20	34
E	12	32	35	20	0	16
F	15	16	19	34	16	0

b AE(12), EF(16), FB(16) BC(10), CD(15), DA(32)
i.e.
A E F B C D A upper bound = 101 km

c In the original network AD is not a direct path. The tour becomes A E F B C D E A

d e.g.

$$\left.\begin{array}{l} \text{B C D E A F B} \\ \text{C B F A E D C} \\ \text{D C B F A E D} \\ \text{E A F B C D E} \\ \text{F A E D C B F} \end{array}\right\} \quad \text{length 88}$$

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Review Exercise 1
Exercise A, Question 4

Question:

A manufacturing company makes 3 products X, Y and Z. The numbers of each product made are x, y and z respectively and £P is the profit. There are two machines which are available for a limited time. These time limitations produce two constraints. In the process of using the simplex algorithm, the following tableau is obtained, where r and s are slack variables.

Basic variable	x	y	z	r	s	Value
y	0	1	$3\frac{1}{3}$	1	$-\frac{1}{3}$	1
x	1	0	-3	-1	$\frac{1}{2}$	3
P	0	0	1	1	1	33

a State how you know that this tableau is optimal (final).

b By writing out the profit equation, or otherwise, explain why a further increase in profit is not possible under these constraints.

c From this tableau, deduce

i the maximum profit,

ii the optimum number of X, Y and Z that should be produced to maximise the profit. ***E***

Solution:

a There are no negative entries in the objective row

b Profit equation

$P + z + r + s = 33$

$P = 33 - (z + r + s)$

At present z, r and s are all zero. If they increase P will decrease. Hence P is maximal

c i $P = 33$

ii $x = 3 \quad y = 1, z = 0$

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Review Exercise 1
Exercise A, Question 5

Question:

Freezy Co. has three factories A, B and C. It supplies freezers to three shops D, E and F. The table shows the transportation cost in pounds of moving one freezer from each factory to each outlet. It also shows the number of freezers available for delivery at each factory and the number of freezers required at each shop. The total number of freezers required is equal to the total number of freezers available.

	D	E	F	Available
A	21	24	16	24
B	18	23	17	32
C	15	19	25	14
Required	20	30	20	

a Use the north-west corner rule to find an initial solution.
b Obtain improvement indices for each unused route.
c Use the stepping-stone method **once** to obtain a better solution and state its cost.

E

Solution:

a

	D	E	F
A	20	4	
B		26	6
C			14

b $S_A = 0$ $\quad S_B = 1$ $\quad S_C = 7$

$D_D = 21$ $\quad D_E = 24$ $\quad D_F = 18$

$I_{AF} = 16 - 0 - 18 = -2$
$I_{BD} = 18 + 1 - 21 = -2$
$I_{CD} = 15 - 7 - 21 = -13$
$I_{CE} = 19 - 7 - 24 = -12$

	D	E	F
A	$20-\theta$	$4+\theta$	
B		$26-\theta$	$6+\theta$
C	θ		$14-\theta$

entering cell CD
$\theta = 14$
exiting cell CF

	D	E	F
A	6	18	
B		12	20
C	14		

cost £1384

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Review Exercise 1
Exercise A, Question 6

Question:

A large room in a hotel is to be prepared for a wedding reception. The tasks that need to be carried out are:

I clean the room,
II arrange the tables and chairs,
III set the places,
IV arrange the decorations.

The tasks need to be completed consecutively and the room must be prepared in the *least possible time*. The tasks are to be assigned to four teams of workers A, B, C and D. Each team must carry out only one task. The table below shows the times, in minutes, that each team takes to carry out each task.

	A	B	C	D
I	17	24	19	18
II	12	23	16	15
III	16	24	21	18
IV	12	24	18	14

a Use the Hungarian algorithm to determine which team should be assigned to each task. You must make your method clear and show
i the state of the table after each stage in the algorithm,
ii the final allocation.
b Obtain the minimum total time taken for the room to be prepared. ***E***

Solution:

a
$$\begin{array}{cccc} 17 & 24 & 19 & 18 \\ 12 & 23 & 16 & 15 \\ 16 & 24 & 21 & 18 \\ 12 & 24 & 18 & 14 \end{array}$$

Reducing rows gives:

$$\begin{array}{cccc} 0 & 7 & 2 & 1 \\ 0 & 11 & 4 & 3 \\ 0 & 8 & 5 & 2 \\ 0 & 12 & 6 & 2 \end{array}$$

Reducing columns gives:

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 5 & 4 & 1 \end{array}$$

No assignment possible as zeroes can all be covered by 2 lines (2 < 4)
Minimum uncovered element is 1
Applying algorithm gives:

$$\begin{array}{cccc} 1 & 0 & 0* & 0 \\ 0* & 3 & 1 & 1 \\ 0 & 0* & 2 & 0 \\ 0 & 4 & 3 & 0* \end{array}$$

Now requires 4 lines to cover all zeroes so assignment now possible
(1, 3) – only zero in column 3
(3, 2) – row 1 already used and now only zero in C2
(4, 4) – only remaining possibility in C4
(2, 1) – must then be used
I – C, II – A, III – B, IV – D

b Time of this assignment
$19 + 12 + 24 + 14 = 69$ minutes

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Review Exercise 1
Exercise A, Question 7

Question:

A three-variable linear programming problem in x, y and z is to be solved. The objective is to maximise the profit P. The following tableau was obtained.

Basic variable	x	y	z	r	s	t	Value
s	3	0	2	0	1	$-\frac{2}{3}$	$\frac{2}{3}$
r	4	0	$\frac{7}{2}$	1	0	8	$\frac{9}{2}$
y	5	1	7	0	0	3	7
P	3	0	2	0	0	8	63

a State, giving your reason, whether this tableau represents the optimal solution.
b State the values of every variable.
c Calculate the profit made on each unit of y. ***E***

Solution:

a *Yes.* there are *no negative* values in the *profit row*
b $P=63, x=0, y=7, z=0, r=\frac{9}{2}, s=\frac{2}{3}, t=0$
c $\frac{63}{7}=9$

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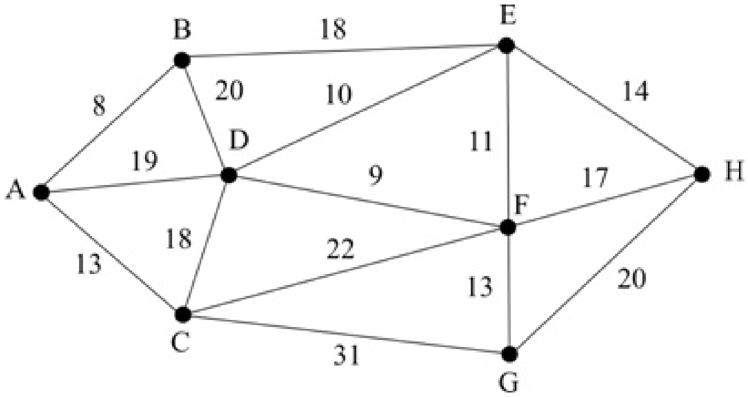
Review Exercise 1
Exercise A, Question 8

Question:

a Explain the difference between the classical and practical travelling salesman problems.



The network above shows the distances, in kilometres, between eight McBurger restaurants. An inspector from head office wishes to visit each restaurant. His route should start and finish at A, visit each restaurant at least once and cover a minimum distance.

b Obtain a minimum spanning tree for the network using Kruskal's algorithm. You should draw your tree and state the order in which the arcs were added.

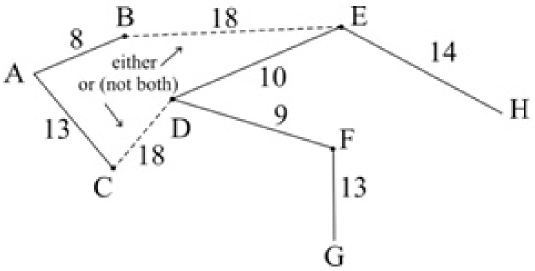
c Use your answer to part **b** to determine an initial upper bound for the length of the route.

d Starting from your initial upper bound and using an appropriate method, find an upper bound which is less than 135 km. State your tour. ***E***

Solution:

a In the *practical* T.S.P each vertex must be visited *at least once*
In the *classical* T.S.P. each vertex must be visited *exactly once*

b AB, DF, DE, (reject EF) $\begin{Bmatrix} \text{FG} \\ \text{AC} \end{Bmatrix}$, EH $\begin{Bmatrix} \text{DC} \\ \text{or} \\ \text{BE} \end{Bmatrix}$



c Initial upper bound $= 2 \times 85 = 170$ km

d When CD is part of tree
Use GH (saving 26) and BD (saving 19) giving a new upper bound of 125 km
Tour A B D E H G F D C A
e.g. when BE is part of tree
Use CG (saving 40) giving a new upper bound of 130 km
Tour A B E H E D F G C A

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Review Exercise 1
Exercise A, Question 9

Question:

In a quiz there are four individual rounds, Art, Literature, Music and Science. A team consists of four people, Donna, Hannah, Kerwin and Thomas. Each of four rounds must be answered by a different team member.
The table shows the number of points that each team member is likely to get on each individual round.

	Art	Literature	Music	Science
Donna	31	24	32	35
Kerwin	19	14	20	21
Hannah	16	10	19	22
Thomas	18	15	21	23

Use the Hungarian algorithm, reducing rows first, to obtain an allocation which maximises the total points likely to be scored in the four rounds. You must make your method clear and show the table after each stage. ***E***

Solution:

Subtract all terms from some $n \geq 35$, eg 35

$$\begin{matrix} 4 & 11 & 3 & 0 \\ 16 & 21 & 15 & 14 \\ 19 & 25 & 16 & 13 \\ 17 & 20 & 14 & 12 \end{matrix}$$

Reducing rows then column $\begin{matrix} 2 & 4 & 2 & 0 \\ 4 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 1 & 1 & 0 \end{matrix}$

minimum uncovered 1 $\begin{matrix} 1 & 3 & 1 & 0 \\ 3 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{matrix}$

minimum uncovered 1 $\begin{matrix} 0 & 2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 1 \end{matrix}$

e.g. matching $\begin{matrix} D-A \\ H-S \\ K-M \\ T-L \end{matrix}$ or $\begin{matrix} A \\ S \\ L \\ M \end{matrix}$ or $\begin{matrix} M \\ S \\ A \\ L \end{matrix}$ or $\begin{matrix} S \\ M \\ A \\ L \end{matrix}$

Total 88 points

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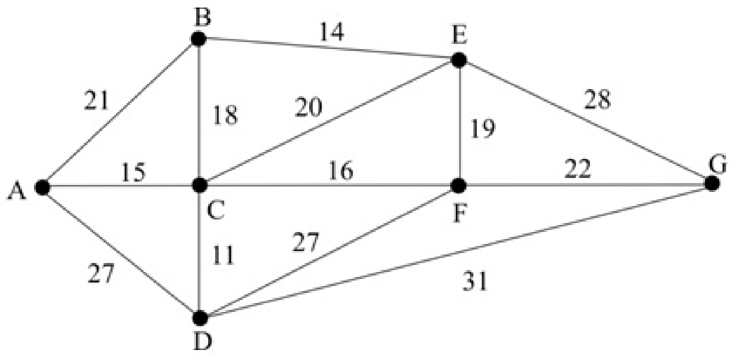
Review Exercise 1
Exercise A, Question 10

Question:



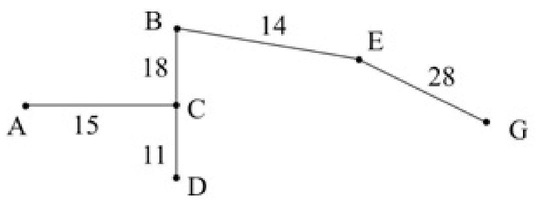
The network above shows the distances, in km, of the cables between seven electricity relay stations A, B, C, D, E, F and G. An inspector needs to visit each relay station. He wishes to travel a minimum distance, and his route must start and finish at the same station.

By deleting C, a lower bound for the length of the route is found to be 129 km.

a Find another lower bound for the length of the route by deleting F. State which is the best lower bound of the two.
b By inspection, complete the table of least distances.
The table can now be taken to represent a complete network.
c Using the nearest-neighbour algorithm, starting at F, obtain an upper bound to the length of the route. State your route. ***E***

Solution:

a Deleting F leaves residual spanning tree



r.s.t. length $= 86$
So lower bound $= 86 + 16 + 19 = 121$
$\therefore$ better lower bound is 129 by deleting C

b Add 33 to BF and FB
Add 31 to DE and ED

c Tour visits each vertex, order correct using table of least distances.
e.g. F C D A B E G F (actual route F C D C A B E G F) upper bound of 138 km

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Review Exercise 1
Exercise A, Question 11

Question:

Three warehouses W, X and Y supply televisions to three supermarkets J, K and L. The table gives the cost, in pounds, of transporting a television from each warehouse to each supermarket. The warehouses have stocks of 34, 57 and 25 televisions respectively, and the supermarkets require 20, 56 and 40 televisions respectively. The total cost of transporting the televisions is to be minimised.

	J	K	L
W	3	6	3
X	5	8	4
Y	2	5	7

Formulate this transportation problem as a linear programming problem. Make clear your decision variables, objective function and constraints. ***E***

Solution:

Let x_{ij} be *number* of unit transported from i to j
when $i \in \{W, X, Y\}$ and $j \in \{J, K, L\}$

Objective minimise $C = 3x_{WJ} + 6x_{WK} + 3x_{WL} +$
$5x_{XJ} + 8x_{XK} + 4x_{XL} +$
$2x_{YJ} + 5x_{YK} + 7x_{YL}$

Subject to $x_{WJ} + x_{WK} + x_{WL} = 34$
$x_{XJ} + x_{XK} + x_{XL} = 57$
$x_{YJ} + x_{YK} + x_{YL} = 25$
$x_{WJ} + x_{XJ} + x_{YJ} = 20$
$x_{WK} + x_{XK} + x_{YK} = 56$
$x_{WL} + x_{XL} + x_{YL} = 40$
$x_{ij} \geq 0 \quad i \in \{W, X, Y\}$ and $j \in \{J, K, L\}$

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Review Exercise 1
Exercise A, Question 12

Question:

A manager wishes to purchase seats for a new cinema. He wishes to buy three types of seat: standard, deluxe and majestic. Let the number of standard, deluxe and majestic seats to be bought be x, y and z respectively.
He decides that the total number of deluxe and majestic seats should be at most half of the number of standard seats.
The number of deluxe seats should be at least 10% and at most 20% of the total number of seats.
The number of majestic seats should be at least half of the number of deluxe seats.
The total number of seats should be at least 250.
Standard, deluxe and majestic seats each cost £20, £26 and £36, respectively.
The manager wishes to minimise the total cost, £C, of the seats.
Formulate this situation as a linear programming problem, simplifying your inequalities so that all coefficients are integers. ***E***

Solution:

$y+z \le \frac{1}{2}x \Rightarrow 2(y+z) \le x$

$y \ge \frac{10}{100}(x+y+z) \Rightarrow x+z \le 9y$

$y \ge \frac{20}{100}(x-y+z) \Rightarrow x+z \ge 4y$

$z \ge \frac{1}{2}y \qquad \Rightarrow 2z \ge y$

$x \ge 0, y \ge 0, z \ge 0$.

$x+y+z \ge 250$

objective function: minimise $C = 20x+26y+36z$

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Review Exercise 1
Exercise A, Question 13

Question:

Talkalot College holds an induction meeting for new students. The meeting consists of four talks: I (Welcome), II (Options and Facilities), III (Study Tips) and IV (Planning for Success). The four department heads, Clive, Julie, Nicky and Steve, deliver one of these talks each. The talks are delivered consecutively and there are no breaks between talks. The meeting starts at 10 a.m. and ends when all four talks have been delivered. The time, in minutes, each department head takes to deliver each talk is given in the table below.

	Talk I	Talk II	Talk III	Talk IV
Clive	12	34	28	16
Julie	13	32	36	12
Nicky	15	32	32	14
Steve	11	33	36	10

a Use the Hungarian algorithm to find the earliest time that the meeting could end. You must make your method clear and show

i the state of the table after each stage in the algorithm.

ii the final allocation.

b Modify the table so it could be used to find the latest time that the meeting could end. (You do not have to find this latest time.) ***E***

Solution:

a i Reduce *rows then columns* giving

$$\begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline C & 0 & 22 & 16 & 4 \\ J & 1 & 20 & 24 & 0 \\ N & 1 & 18 & 18 & 0 \\ S & 1 & 23 & 26 & 0 \end{array} \quad \text{then} \quad \begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline C & 0 & 4 & 0 & 4 \\ J & 1 & 2 & 8 & 0 \\ N & 1 & 0 & 2 & 0 \\ S & 1 & 5 & 10 & 0 \end{array}$$

3 lines only needed least element 1

or least element

$$\begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline C & 0 & 4 & 0 & 5 \\ J & 0 & 1 & 7 & 0 \\ N & 1 & 0 & 2 & 1 \\ S & 0 & 4 & 9 & 0 \end{array} \quad \text{or} \quad \begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline C & 0 & 5 & 0 & 5 \\ J & 0 & 2 & 7 & 0 \\ N & 0 & 0 & 1 & 0 \\ S & 0 & 5 & 9 & 0 \end{array}$$

or *columns then rows* giving

$$\begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline C & 1 & 2 & 0 & 6 \\ J & 2 & 0 & 8 & 2 \\ N & 4 & 0 & 4 & 4 \\ S & 0 & 1 & 8 & 0 \end{array}$$

(then no change)

3 lines only needed ⊥ and *either* row 1 *or* column 3

if row 1 least uncovered element is 2

$$\begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline C & 1 & 4 & 0 & 6 \\ J & 0 & 0 & 6 & 0 \\ N & 2 & 0 & 2 & 2 \\ S & 0 & 3 & 8 & 0 \end{array}$$

if column 3 least uncovered element is 1

$$\begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline C & 0 & 2 & 0 & 5 \\ J & 1 & 0 & 8 & 1 \\ N & 3 & 0 & 4 & 3 \\ S & 0 & 2 & 9 & 0 \end{array}$$

then I least uncovered element 1

$$\begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline \text{C} & 0 & 3 & 0 & 5 \\ \text{J} & 0 & 0 & 7 & 0 \\ \text{N} & 2 & 0 & 3 & 2 \\ \text{S} & 0 & 3 & 9 & 0 \end{array}$$

ii C - III, J - I or IV, N - II, S - IV or I
83 minutes $\therefore$ 11:23 am

b Subtracting all entries from some $n \geq 36$
e.g. subtracting from 36

	I	II	III	IV
C	24	2	8	20
J	23	4	0	24
N	21	4	4	22
S	25	3	0	26

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Review Exercise 1
Exercise A, Question 14

Question:

The table shows the least distances, in km, between five towns, A, B, C, D and E.

	A	B	C	D	E
A	–	153	98	124	115
B	153	–	74	131	149
C	98	74	–	82	103
D	124	131	82	–	134
E	115	149	103	134	–

Nassim wishes to find an interval which contains the solution to the travelling salesman problem for this network.

a Making your method clear, find an initial upper bound starting at A and using
  i the minimum spanning tree method,
  ii the nearest neighbour algorithm.
b By deleting E, find a lower bound.
c Using your answers to parts **a** and **b**, state the smallest interval that Nassim could correctly write down. ***E***

Solution:

a i Minimum connector using Prim: AC, CB, CD, CE
$\text{length} = 98 + 74 + 82 + 103 = 357 \quad \{1, 3, 2, 4, 5\}$
So upper bound $= 2 \times 357 = 714$

ii A$\underline{(98)}$ C$\underline{(74)}$ B$\underline{(131)}$ D$\underline{(134)}$ E$\underline{(115)}$A
$\text{length} = 98 + 74 + 131 + 134 + 115 = 552$

b Residual minimum connector is AC, CB, CD length 254
Lower bound $= 254 + 103 + 115 = 472$

c $472 \leq \text{solution} \leq 552$

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Review Exercise 1
Exercise A, Question 15

Question:

Three depots, F, G and H, supply petrol to three service stations, S, T and U. The table gives the cost, in pounds, of transporting 1000 litres of petrol from each depot to each service station.

	S	T	U
F	23	31	46
G	35	38	51
H	41	50	63

F, G and H have stocks of 540 000, 789 000 and 673 000 litres respectively. S, T and U require 257 000, 348 000 and 410 000 litres respectively. The total cost of transporting the petrol is to be minimised.

Formulate this problem as a linear programming problem. Make clear your decision variables, objective function and constraints. ***E***

Solution:

Let x_{ij} be the *number* of units transported from i to j, in 1000 litres where $i \in \{\text{F}, \text{G}, \text{H}\}$ and $j \in \{\text{S}, \text{T}, \text{U}\}$

$$\text{minimise } C = 23x_{\text{fs}} + 31x_{\text{ft}} + 46x_{\text{fu}} + 35x_{\text{gs}} + 38x_{\text{gt}} + 51x_{\text{gu}} + 41x_{\text{hs}} + 50x_{\text{ht}} + 63x_{\text{hu}}$$

unbalanced

subject to
$$x_{\text{fs}} + x_{\text{ft}} + x_{\text{fu}} \le 540$$
$$x_{\text{gs}} + x_{\text{gt}} + x_{\text{gu}} \le 789$$
$$x_{\text{hs}} + x_{\text{ht}} + x_{\text{hu}} \le 673$$
$$\left.\begin{aligned} x_{\text{fs}} + x_{\text{gs}} + x_{\text{hs}} &\le 257 \\ x_{\text{ft}} + x_{\text{gt}} + x_{\text{ht}} &\le 348 \\ x_{\text{fu}} + x_{\text{gu}} + x_{\text{hu}} &\le 410 \end{aligned}\right\} \text{accept} = \text{here}$$

$$x_{ij} \ge 0$$

Accept introduction of a dummy demand methods.

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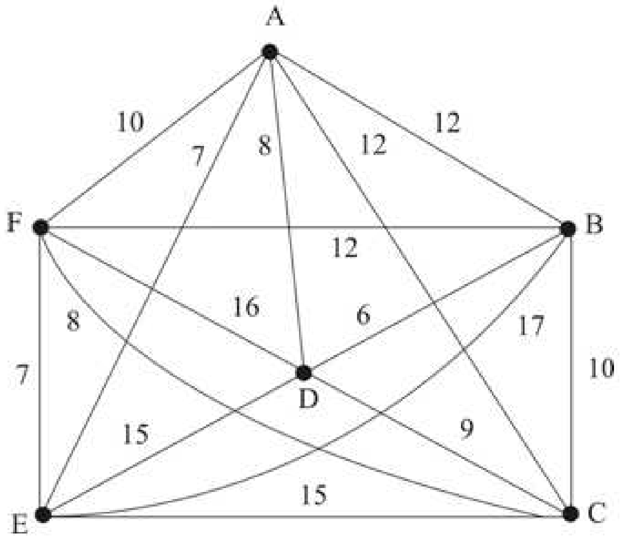
Review Exercise 1
Exercise A, Question 16

Question:



The diagram shows six towns A, B, C, D, E and F and the roads joining them. The number on each arc gives the length of that road in miles.

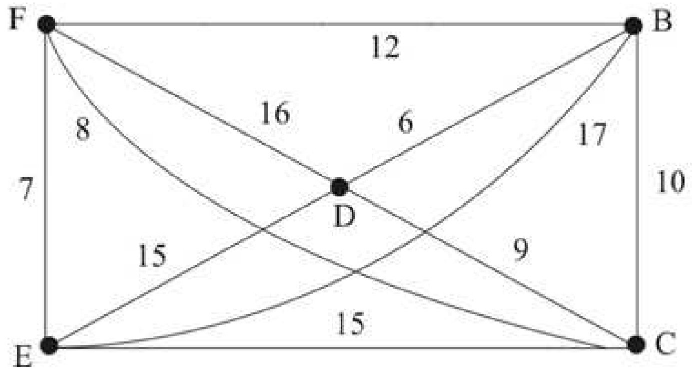
a By deleting vertex A, obtain a lower bound for the solution to the travelling salesman problem.

The nearest neighbour algorithm for finding a possible salesman tour is as follows:
Step 1: Let V be the current vertex.
Step 2: Find the nearest unvisited vertex to the current vertex, move directly to that vertex and call it the current vertex.
Step 3: Repeat step 2 until all vertices have been visited and then return directly to the start vertex.

b **i** Use this algorithm to find a tour starting at the vertex A. State clearly the tour and give its length.
ii Starting at an appropriate vertex, use the algorithm to find a tour of shorter length. ***E***

Solution:

a Deleting vertex A we obtain



By Kruskal's algorithm an MST is DB(6), EF(7), CF(8), DC(9) of weight 30

The two edges of least weight at A are AE(7) and AD(8)

$\therefore$ A lower bound is $30+8+7=45$

b **i** A – nearest neighbour E(7)
E – nearest neighbour F(7)
F – nearest neighbour C(8)
C – nearest neighbour D(9)
D – nearest neighbour B(6)

Complete tour with BA(12)
A E F C D B A – length 49

ii Choose a tour that does not use AB
e.g. DB(6) BC(10), CF(8), FE(4), EA(4)
Complete with AD(8), D B C F E A D.
Total weight 46

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Review Exercise 1
Exercise A, Question 17

Question:

Warehouse / Factory	W_1	W_2	W_3	Availabilities
F_1	7	8	6	4
F_2	9	2	4	3
F_3	5	6	3	8
Requirements	2	9	4	

A manufacturer has 3 factories F_1, F_2, F_3 and 3 warehouses W_1, W_2, W_3. The table shows the cost C_{ij}, in appropriate units, of sending one unit of product from factory F_i to warehouse W_j. Also shown in the table are the number of units available at each factory F_i and the number of units required at each warehouse W_j. The total number of units available is equal to the number of units required.

a Use the north-west corner rule to obtain a possible pattern of distribution and find its cost.

b Calculate shadow costs R_i and K_j for this pattern and hence obtain improvement indices I_{ij} for each route.

c Using your answer to part **b**, explain why the pattern is optimal. ***E***

Solution:

a

	W_1	W_2	W_3	Available
F_1	2	2		4
F_2		3		3
F_3		4	4	8
Require	2	9	4	

Cost $2\times7+2\times8+3\times2+4\times6+4\times3=14+16+6+24+12=72$

b For occupied cells $R_1+K_2=C_3$ gives

$(1,1)R_1+K_1 = 7j(1.2)R_1+K_2=8: (2.2)R_2-K_2=2$

$(3,2)R_3+K_2 = 6j(3,3)R_3+K_3=3$

Taking $R_1=0$ we obtain $K_1=7, K_2=8, R_2=-6, R_3=-2, K_3=5$

Shadow costs		7	8	5	
		W_1	W_2	W_3	
0	F_1	7	8		4
−6	F_2		2		3
−2	F_3		6	3	8
		2	9	4	

$F_1=0$ $W_1=7$
$F_2=-6$ $W_2=8$
$F_3=-2$ $W_3=5$

Improvement indices $I_{ij}=C_{ij}-R_i-K_j$

$I_{13}=6-5-0=1$

$I_{21}=9-7-(-6)=8$

$I_{23}=4-5-(-6)=5$

$I_{31}=5-7-(-2)=0$

c No negative improvement indices and so given solution is optimal and gives minimum cost. If there was a negative I_{ij} then using this route would reduce cost.

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Review Exercise 1
Exercise A, Question 18

Question:

a State the circumstances under which it is necessary to use the simplex algorithm, rather than a graphical method.

The tableau given below arose after one complete iteration of the simplex algorithm.

Basic variable	x	y	z	r	s	Value
y	$\frac{4}{5}$	1	$\frac{2}{5}$	$\frac{1}{5}$	0	$429\frac{2}{5}$
s	$2\frac{1}{5}$	0	$5\frac{3}{5}$	$-\frac{1}{5}$	1	$1243\frac{3}{5}$
P	$-\frac{3}{5}$	0	$-\frac{4}{5}$	$1\frac{3}{5}$	0	$3435\frac{1}{5}$

b State the column that was used as the pivotal column for the first iteration.
c Perform one further complete iteration to obtain the next complete tableau.
d State the values of P, x, y and z displayed by your tableau in part **c**.
e State, giving a reason, whether your values in part **d** give the optimal solution. ***E***

Solution:

a If the number of variables ≥ 3 use simplex

b Column y

c

b.v.	x	y	z	r	s	numbers	
y	$\frac{9}{14}$	1	0	$\frac{2}{7}$	$\frac{-1}{14}$	$340\frac{4}{7}$	$\text{R}1-\frac{2}{5}\text{R}3$
z	$\frac{11}{28}$	0	1	$\frac{-3}{14}$	$\frac{5}{28}$	$222\frac{1}{14}$	$\text{R}2+5\frac{3}{5}$
P	$-\frac{2}{7}$	0	0	$1\frac{3}{7}$	$\frac{1}{7}$	$3612\frac{6}{7}$	$\text{R}3+\frac{4}{5}\text{R}3$

d $P = 3612\frac{6}{7}$ $x = 0$ $y = 340\frac{4}{7}$ $z = 222\frac{1}{14}$

e No. bottom row still contains a negative, x can be increased.

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Review Exercise 1
Exercise A, Question 19

Question:

An engineering company has 4 machines available and 4 jobs to be completed. Each machine is to be assigned to one job. The time, in hours, required by each machine to complete each job is shown in the table below.

	Job 1	**Job 2**	**Job 3**	**Job 4**
Machine 1	14	5	8	7
Machine 2	2	12	6	5
Machine 3	7	8	3	9
Machine 4	2	4	6	10

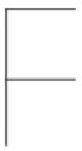
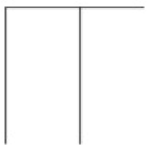
Use the Hungarian algorithm, *reducing rows first*, to obtain the allocation of machines to jobs which minimises the total time required. State this minimum time. ***E***

Solution:

a Reducing rows

$$\begin{array}{cccc} 9 & 0 & 3 & 2 \\ 0 & 10 & 4 & 3 \\ 4 & 5 & 0 & 6 \\ 0 & 2 & 4 & 8 \end{array} \xrightarrow[\text{columns}]{\text{reducing}} \begin{array}{cccc} 9 & 0 & 3 & 0 \\ 0 & 10 & 4 & 1 \\ 4 & 5 & 0 & 4 \\ 0 & 2 & 4 & 6 \end{array}$$

b Testing for optimality – 3 lines are enough


or


Minimum uncovered element is 1

$$\begin{array}{cccc} 10 & 0 & 3 & 0 \\ 0 & 9 & 3 & 0 \\ 5 & 5 & 0 & 4 \\ 0 & 1 & 3 & 5 \end{array} \quad \text{or} \quad \begin{array}{cccc} 10 & 0 & 4 & 0 \\ 0 & 9 & 4 & 0 \\ 4 & 4 & 0 & 3 \\ 0 & 1 & 4 & 5 \end{array}$$

4 lines now needed

c Final matching
Machine 1 - Job 2 (5)
Machine 2 - Job 4 (5)
Machine 3 - Job 3 (3)
Machine 4 - Job 1 (2)
Minimum time: 15 hours

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Review Exercise 1
Exercise A, Question 20

Question:

The following minimising transportation problem is to be solved.

	J	K	Supply
A	12	15	9
B	8	17	13
C	4	9	12
Demand	9	11	

a Complete the first table on the worksheet.
b Explain why an extra demand column was added to the table.

A possible north-west corner solution is:

	J	K	L
A	9	0	
B		11	2
C			12

c Explain why it was necessary to place a zero in the first row of the second column.

After three iterations of the stepping-stone method the table becomes:

	J	K	L
A		8	1
B			13
C	9	3	

d Taking the most negative improvement index as the entering square for the stepping-stone method, solve the transportation problem. You must make your shadow costs and improvement indices clear and demonstrate that your solution is optimal. ***E***

Solution:

a Adds zero for cost in third column
Adds 14 as the demand value

b The total supply is greater than the total demand

c The solution would otherwise be degenerate

d

		10	15	0
		J	K	L
0	A		8	1
0	B			13
−6	C	9	3	

$I_{AJ} = 12 - 0 - 10 = 2$
$I_{BJ} = 8 - 0 - 10 = -2^*$
$I_{BC} = 17 - 0 - 15 = 2$
$I_{CL} = 0 + 6 - 0 = 6$

	J	K	L
A		$8-\theta$	$1+\theta$
B	θ		$13-\theta$
C	$9-\theta$	$3+\theta$	

$\theta = 8$
Entering square BJ
Exiting square AK

		8	13	0
		J	K	L
0	A			9
0	B	8		5
−4	C	1	11	

$I_{AJ} = 12 - 0 - 3 = 4$
$I_{AK} = 15 - 0 - 13 = 2$
$I_{BK} = 17 - 0 - 13 = 4$
$I_{CL} = 0 + 4 - 0 = 4$
No negatives, so optimal

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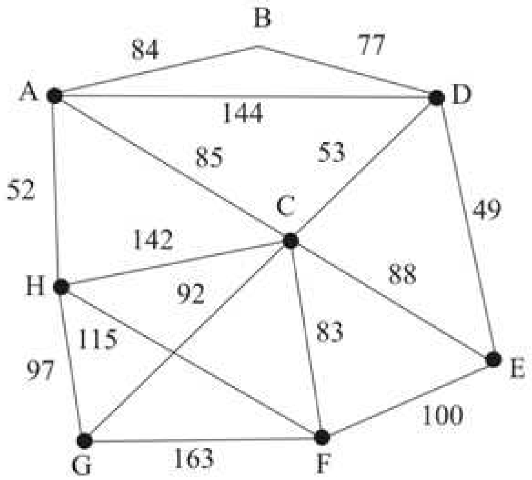
Review Exercise 1
Exercise A, Question 21

Question:



The network above shows the distances in km, along the roads between eight towns, A, B, C, D, E, F, G and H. Keith has a shop in each town and needs to visit each one. He wishes to travel a minimum distance and his route should start and finish at A.

By deleting D, a lower bound for the length of the route was found to be 586 km.
By deleting F, a lower bound for the length of the route was found to be 590 km.

a By deleting C, find another lower bound for the length of the route. State which is the best lower bound of the three, giving a reason for your answer.

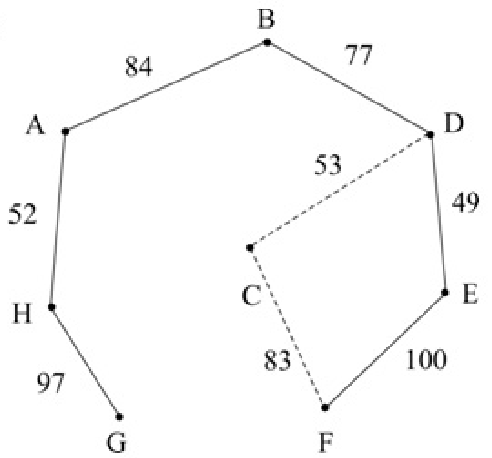
b By inspection complete the table of least distances.

The table can now be taken to represent a complete network.
The nearest neighbour algorithm was used to obtain upper bounds for the length of the route:
Starting at D, an upper bound for the length of the route was found to be 838 km.
Starting at F, an upper bound for the length of the route was found to be 707 km.

c Starting at C, use the nearest neighbour algorithm to obtain another upper bound for the length of the route. State which is the best upper bound of the three, giving a reason for your answer. ***E***

Solution:

a



R.M.S.T
e.g. AH, AB, BD, DE, HG, EF
using Prim's

length of R.M.S.T = 459

$\therefore$ lower bound $= 459 + 53 + 83 = 595$ km (deleting C)

Best lower bound is 595 km, by deleting C as it is the highest lower bound found.

b Adds 167 to AF and FA
137 to CH and HC
136 to DF and FD
145 to DG and GD

c $C_{53}\ D_{49}\ E_{120}\ F_{115}\ H_{52}\ A_{84}\ B_{222}\ G_{92}\ C$

Upper bound, starting at $C = 767$ km

$\therefore$ Best upper bound is 707 starting at F as it is the lowest upper bound found.

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Review Exercise 1
Exercise A, Question 22

Question:

a Describe a practical problem that could be solved using the transportation algorithm.

A problem is to be solved using the transportation problem. The costs are shown in the table. The supply is from A, B and C and the demand is at d and e.

	d	e	Supply
A	5	3	45
B	4	6	35
C	2	4	40
Demand	50	60	

b Explain why it is necessary to add a third demand f.

c Use the north-west corner rule to obtain a possible pattern of distribution and find its cost.

d Calculate shadow costs and improvement indices for this pattern.

e Use the stepping-stone method once to obtain an improved solution and its cost. ***E***

Solution:

a Idea of many supply and demand points and many units to be moved. Costs are variable and dependent upon the supply and demand points, need to minimise costs. *Practical*

b Supply $= 120$ Demand $= 110$ so not balanced

c Adds 0, 0, 0, 10 to column f

	d	e	f
A	45		
B	5	30	
C		30	10

Cost 545

d $R_1 = 0$ $R_2 = -1$ $R_3 = -3$
$K_1 = 5$ $K_2 = 7$ $K_3 = 3$

Shadow costs

$Ae = 3 - 0 - 7 = -4 \leftarrow$
$Af = 0 - 0 - 3 = -3$
$Bf = 0 + 1 - 3 = -2$
$Cd = 2 + 3 - 5 = 0$

Improvement indices

e $Ae^+ \rightarrow Be^- \rightarrow Bd^+ \rightarrow Ad^-$ so $\theta = 30$

	d	e	f
A	15	30	
B	35		
C		30	10

Cost 425

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Review Exercise 1
Exercise A, Question 23

Question:

Four salespersons Ann, Brenda, Connor and Dave are to be sent to visit four companies 1, 2, 3 and 4. Each salesperson will visit exactly one company, and all companies will be visited.
Previous sales figures show that each salesperson will make sales of different values, depending on the company that they visit. These values (in £10 000s) are shown in the table below.

	1	2	3	4
Ann	26	30	30	30
Brenda	30	23	26	29
Connor	30	25	27	24
Dave	30	27	25	21

a Use the Hungarian algorithm to obtain an allocation that **maximises** the sales. You must make your method clear and show the table after each stage.
b State the value of the maximum sales.
c Show that there is a second allocation that maximises the sales. ***E***

Solution:

a To maximise, subtract all entries from $n \geq 30$

e.g. $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 7 & 4 & 1 \\ 0 & 5 & 3 & 6 \\ 0 & 3 & 5 & 9 \end{bmatrix}$

minimum uncovered element is 1

so $\begin{bmatrix} & & & \\ 0 & 6 & 3 & 0 \\ 0 & 4 & 2 & 5 \\ 0 & 2 & 4 & 8 \end{bmatrix}$

or

minimum element is 2

minimum element is 2

$\begin{bmatrix} 7 & 0 & 0 & 2 \\ 0 & 4 & 1 & 0 \\ 0 & 2 & 0 & 5 \\ 0 & 0 & 2 & 8 \end{bmatrix}$ $\begin{bmatrix} 7 & 0 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 2 & 6 \end{bmatrix}$

$A-2\ B-4\ C-3\ D-1$

or $A-3\ B-4\ C-1\ D-2$

b £1 160 000

c Gives other solution from part **a**.

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Review Exercise 1
Exercise A, Question 24

Question:

The manager of a car hire firm has to arrange to move cars from three garages A, B and C to three airports D, E and F so that customers can collect them. The table below shows the transportation cost of moving one car from each garage to each airport. It also shows the number of cars available in each garage and the number of cars required at each airport. The total number of cars available is equal to the total number required.

	Airport D	Airport E	Airport F	Cars available
Garage A	£20	£40	£10	6
Garage B	£20	£30	£40	5
Garage C	£10	£20	£30	8
Cars required	6	9	4	

a Use the north-west corner rule to obtain a possible pattern of distribution and find its cost.

b Calculate shadow costs for this pattern and hence obtain improvement indices for each route.

c Use the stepping-stone method to obtain an optimal solution and state its cost. ***E***

Solution:

a *e.g.*

	D	E	F
A	6		
B	0	5	
C		4	4

Cost £470

or

	D	E	F
A	6	0	
B		5	
C		4	4

b $S_A = 0 \quad S_B = 0 \quad S_C = -10$

$D_D = 20 \quad D_E = 30 \quad D_F = 40$

$I_{AE} = 40 - 30 = 10$

$I_{AF} = 10 - 40 = -30$

$I_{BF} = 40 - 40 = 0$

$I_{CD} = 10 - 10 = 0$

or

$S_A = 0 \quad S_B = -10 \quad S_C = -20$

$D_D = 20 \quad D_E = 40 \quad D_F = 50$

$I_{AF} = 10 - 50 = -40$

$I_{BD} = 20 - 10 = 10$

$I_{BF} = 40 - 40 = 0$

$I_{CD} = 10 - 0 = 10$

c Choose A F as entering route

$AF(+) \to CF(-) \to CE(+) \to BE(-) \to BD(+) \to AD(-)$

Exiting route CF $\theta = 4$

	D	E	F
A	2		4
B	4	1	
C		8	

$S_A = 0 \quad S_B = 0 \quad S_C = -10$

$D_D = 20 \quad D_E = 30 \quad D_F = 10$

$I_{AE} = 10, I_{BF} = 30,$

$I_{CD} = 0 \quad I_{CF} = 30$

$\therefore$ optimal

cost = £350

or

$AF(+) \to CF(-) \to CE(+) \to AE(-)$

Exiting route AE $\theta = 0$

	D	E	F
A	6		0
B		5	
C		4	4

So $S_A = 0 \quad S_B = 30 \quad S_C = 20$

$D_D = 20 \quad D_E = 0 \quad D_F = 10$

$I_{AE} = 40, I_{BD} = -30, I_{BF} = 0, I_{CD} = -30$

e.g. $CD(+) \to AD(-) \to AF(+) \to CF(-)$ $\theta = 4$

	D	E	F
A	2		4
B		5	
C	4	4	

$S_A = 0 \quad S_B = 0 \quad S_C = -10$

$D_D = 20 \quad D_E = 30 \quad D_F = 10$

$I_{AE} = 10, I_{BD} = 0 \quad I_{BF} = 30 \quad I_{CF} = 30$

$\therefore$ optimal cost £350

or $DB(+) \to BE(-) \to CE(+) \to CD(-)$ $\theta = 4$

giving left hand solution table

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Review Exercise 1
Exercise A, Question 25

Question:

A chemical company makes 3 products X, Y and Z. It wishes to maximise its profit £P. The manager considers the limitations on the raw materials available and models the situation with the following linear programming problem.

Maximise $P = 3x + 6y + 4z$,
subject to $x + z \le 4$,
$x + 4y + 2z \le 6$,
$x + y + 2z \le 12$,
$x \ge 0, y \ge 0, z \ge 0$,

where x, y and z are the weights, in kg, of products X, Y and Z respectively.

A possible tableau is

Basic variable	x	y	z	r	s	t	Value
r	1	0	1	1	0	0	4
s	1	4	2	0	1	0	6
t	1	1	2	0	0	1	12
P	−3	−6	−4	0	0	0	0

a Explain
  i the purpose of the variables r, s and t,
  ii the final row of the tableau.
b Solve this linear programming problem by using the simplex algorithm.
Increase y for your first iteration and then increase x for your second iteration.
c Interpret your solution. ***E***

Solution:

a i Slack variables used to enable us to write inequalities as equalities. All slack variable are ≥ 0

ii $P - 3x - 6y - 4z = 0$

b

b.v.	x	y	z	r	s	t	value	row ops
r	1	0	1	1	0	0	4	
s	1	4	2	0	1	0	6	
t	1	1	2	0	0	1	12	
P	-3	-6	-4	0	0	0	0	

b.v.	x	y	z	r	s	t	value	row ops
r	1	0	1	1	0	0	4	No change
Y	$\frac{1}{4}$	1	$\frac{1}{2}$	0	$\frac{1}{4}$	0	$1\frac{1}{2}$	R2 $\div$ 4
t	$\frac{3}{4}$	0	$1\frac{1}{2}$	0	$-\frac{1}{4}$	1	$10\frac{1}{2}$	R3 $-$ R2
P	$-1\frac{1}{2}$	0	-1	0	$1\frac{1}{2}$	0	9	R4 + 6R2

b.v.	x	y	z	r	s	t	value	row ops
x	1	0	1	1	0	0	4	R1 $\div$ 1
y	0	1	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	R2 $-\frac{1}{4}$R1
t	0	0	$\frac{3}{4}$	$-\frac{3}{4}$	$-\frac{1}{4}$	1	$7\frac{1}{2}$	R3 $-\frac{3}{4}$R1
P	0	0	$\frac{1}{2}$	$1\frac{1}{2}$	$1\frac{1}{2}$	0	15	R4 $+1\frac{1}{2}$R1

c Maximum profit is £15

when $x = 4$ kg, $y = \frac{1}{2}$ kg, $z = 0$ kg

The first and second constraints have no slack

There is a slack of $7\frac{1}{2}$ in the third constraint.

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Review Exercise 1
Exercise A, Question 26

Question:

The table below shows the distances, in km, between six towns A, B, C, D, E and F.

	A	B	C	D	E	F
A	–	85	110	175	108	100
B	85	–	38	175	160	93
C	110	38	–	148	156	73
D	175	175	148	–	110	84
E	108	160	156	110	–	92
F	100	93	73	84	92	–

a Starting from A, use Prim's algorithm to find a minimum connector and draw the minimum spanning tree. You must make your method clear by stating the order in which the arcs are selected.

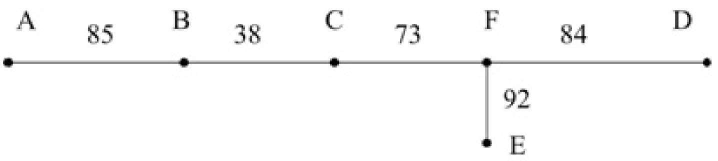
b **i** Using your answer to part **a** obtain an initial upper bound for the solution of the travelling salesman problem.

ii Use a short cut to reduce the upper bound to a value less than 680.

c Starting by deleting F, find a lower bound for the solution of the travelling salesman problem. ***E***

Solution:

a Order of arcs: AB, BC, CF, FD, FE



b **i** $2 \times 372 = 744$

ii e.g. AD saves 105 giving 639
or AE saves 180 giving 564
AF saves 96 giving 648
DE saves 66 giving 678

c Residual M.S.T.
AB, BC, AE, ED

Lower bound $= 341 + 73 + 84$
$= 498$

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Review Exercise 1
Exercise A, Question 27

Question:

Flatland UK Ltd makes three types of carpet, the Lincoln, the Norfolk and the Suffolk. The carpets all require units of black, green and red wool.
For each roll of carpet,
the Lincoln requires 1 unit of black, 1 of green and 3 of red,
the Norfolk requires 1 unit of black, 2 of green and 2 of red,
and the Suffolk requires 2 units of black, 1 of green and 1 of red.

There are up to 30 units of black, 40 units of green and 50 units of red available each day. Profits of £50 , £80 and £60 are made on each roll of Lincoln, Norfolk and Suffolk respectively.
Flatland UK Ltd wishes to maximise its profit.
Let the number of rolls of the Lincoln, Norfolk and Suffolk made daily be x, y and z respectively.

a Formulate the above situation as a linear programming problem, listing clearly the constraint as inequalities in their simplest form, and stating the objective function.
This problem is to be solved using the simplex algorithm. The most negative number in the profit row is taken to indicate the pivot column at each stage.

b Stating your row operations, show that after one complete iteration the tableau becomes

Basic variable	x	y	z	r	s	t	Value
r	$\frac{1}{2}$	0	$1\frac{1}{2}$	1	$-\frac{1}{2}$	0	10
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	20
t	2	0	0	0	-1	1	10
P	-10	0	-20	0	40	0	1600

c Explain the practical meaning of the value 10 in the top row.
d i Perform one further complete iteration of the simplex algorithm.
ii State whether your answer to part **d i** is optimal. Give a reason for your answer.
iii Interpret your current tableau, giving the value of each variable. ***E***

Solution:

a Maximise $P = 50x + 80y + 60z$

Subject to $x + y + 2z \le 30$

$x + 2y + z \le 40$

$3x + 2y + z \le 50$

where $x, y, z \ge 0$

b Initialising tableau

b.v.	x	y	z	r	s	t	value
r	1	1	2	1	0	0	30
s	1	2	1	0	1	0	40
t	3	2	1	0	0	1	50
P	−50	−80	−60	0	0	0	0

Chooses correct pivot, divide R2 by 2

State correct row operation R1 − R2, R3 − 2R2, R4 + 80R2, R2 ÷ 2

c The solution found after one iteration has a slack of 10 units of black per day

d i

b.v.	x	y	z	r	s	t	value
r	$\frac{1}{2}$	0	$\frac{3}{2}$	1	$\frac{-1}{2}$	0	10
y	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	20
t	2	0	0	0	−1	1	10
P	−10	0	−20	0	40	0	1600

(given)

b.v.	x	y	z	r	s	t	value	
z	$\frac{1}{3}$	0	1	$\frac{2}{3}$	$\frac{-1}{3}$	0	$6\frac{2}{3}$	$\text{R}1 \div \frac{3}{2}$
y	$\frac{1}{3}$	1	0	$\frac{-1}{3}$	$\frac{2}{3}$	0	$16\frac{2}{3}$	$\text{R}2 - \frac{1}{2}\text{R}1$
t	2	0	0	0	−1	1	10	R3 − no change
P	$-3\frac{1}{3}$	0	0	$13\frac{1}{3}$	$33\frac{1}{3}$	0	$1733\frac{1}{3}$.	R4+20R1

ii Not optimal, a negative value in profit row

iii $x = 0 \quad y = 16\frac{2}{3} \quad z = 6\frac{2}{3}$

$P = £1733.33 \quad r = 0, s = 0, t = 10$

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Review Exercise 1
Exercise A, Question 28

Question:

	A	B	C	D	E	F
A	–	113	53	54	87	68
B	113	–	87	123	38	100
C	53	87	–	106	58	103
D	54	123	106	–	140	48
E	87	38	58	140	–	105
F	68	100	103	48	105	–

The table shows the distances, in km, between six towns A, B, C, D, E and F.

a Starting from A, use Prim's algorithm to find a minimum connector and draw the minimum spanning tree. You must make your method clear by stating the order in which the arcs are selected.

b **i** Hence form an initial upper bound for the solution to the travelling salesman problem.

ii Use a short cut to reduce the upper bound to a value below 360.

c By deleting A, find a lower bound for the solution to the travelling salesman problem.

d Use your answers to parts **b** and **c** to make a comment on the value of the optimal solution.

e Draw a diagram to show your best route. ***E***

Solution:

a AC (–53), AD(54), DF(–48), CE(–58), EB(–38)

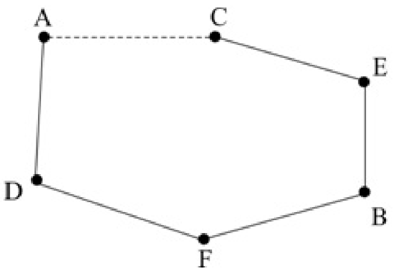
b **i** M.S.T. XZ = 251 × 2 = 502

ii Finding a shortcut to below 360, e.g. FB leaves 351

c M.S.T. is DF, CE, EB, FB length 244
The 2 shortest arcs are AC (–53) and AD (–54) giving a total of 351

d The optimal solution is 351 and is A–C–E–B–F–D–A

e



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Review Exercise 1
Exercise A, Question 29

Question:

Polly has a bird food stall at the local market. Each week she makes and sells three types of packs A, B and C.

Pack A contains 4 kg of bird seed, 2 suet blocks and 1 kg of peanuts.
Pack B contains 5 kg of bird seed, 1 suet block and 2 kg of peanuts.
Pack C contains 10 kg of bird seed, 4 suet blocks and 3 kg of peanuts.

Each week Polly has 140 kg of bird seed, 60 suet blocks and 60 kg of peanuts available for the packs.

The profit made on each pack of A, B and C sold is £3.50, £3.50 and £6.50 respectively. Polly sells every pack on her stall and wishes to maximise her profit, P pence.

Let x, y and z be the numbers of packs of A, B and C sold each week.
An initial simplex tableau for the above situation is

Basic variable	x	y	z	r	s	t	Value
r	4	5	10	1	0	0	140
s	2	1	4	0	1	0	60
t	1	2	3	0	0	1	60
P	−350	−350	−650	0	0	0	0

a Explain the meaning of the variables r, s and t in the context of this question.
b Perform one complete iteration of the simplex algorithm, to form a new tableau T.
Take the most negative number in the profit row to indicate the pivotal column.
c State the value of every variable as given by tableau T.
d Write down the profit equation given by tableau T.
e Use your profit equation to explain why tableau T is not optimal.
Taking the most negative number in the profit row to indicate the pivotal column,
f identify clearly the location of the next pivotal element. ***E***

Solution:

a r, s and t are unused amounts of bird seed (in kg), suet blocks and peanuts (in kg) that Polly has at the end of each week after she has made up and sold her packs.

b

b.v.	x	y	z	r	s	t	value	
z	$\frac{2}{5}$	$\frac{1}{2}$	1	$\frac{1}{10}$	0	0	14	R1÷10
s	$\frac{2}{5}$	-1	0	$\frac{-2}{5}$	1	0	4	R2−4R1
t	$-\frac{1}{5}$	$\frac{1}{2}$	0	$-\frac{3}{10}$	0	1	18	R3−3R1
P	-90	-25	0	65	0	0	9100	R4+650R1

c $x=0$ $y=0$ $z=14$ $r=0$ $s=4$ $t=18$ $P=£91$

d $P-90x-25y+65r=9100$

e $P=9100+90x+25y-65r$
So increasing x or y would increase the profit

f The $\frac{2}{5}$ in the x column and 2nd (s) row.

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Review Exercise 1
Exercise A, Question 30

Question:

A steel manufacturer has 3 factories F_1, F_2 and F_3 which can produce 35, 25 and 15 kilotonnes of steel per year, respectively. Three businesses B_1, B_2 and B_3 have annual requirements of 20, 25 and 30 kilotonnes respectively. The table below shows the cost C_{ij}, in appropriate units, of transporting one kilotonne of steel from factory F_i to business B_j.

			Business	
		B_1	B_2	B_3
Factory	F_1	10	4	11
	F_2	12	5	8
	F_3	9	6	7

The manufacturer wishes to transport the steel to the businesses at minimum total cost.

a Write down the transportation pattern obtained by using the north-west corner rule.

b Calculate all of the improvement indices I_{ij}, and hence show that this pattern is not optimal.

c Use the stepping-stone method to obtain an improved solution.

d Show that the transportation pattern obtained in part **c** is optimal and find its cost.

E

Solution:

a

	B_1	B_2	B_3
F_1	20	15	
F_2		10	15
F_3			15

b

$S(F_1) = 0 \quad S(F_2) = 1 \quad S(F_3) = 0$

$D(B_1) = 10 \quad D(B_2) = 4 \quad D(B_3) = 7$

$I_{13} = 11 - 0 - 7 = 4$

$I_{21} = 12 - 1 - 10 = 1$

$I_{31} = 9 - 0 - 10 = -1$

$I_{32} = 6 - 0 - 4 = 2$

Since I_{31} is negative pattern is not optimal

c

	B_1	B_2	B_3
F_1	$20-\theta$	$15+\theta$	
F_2		$10-\theta$	$15+\theta$
F_3	θ		$15-\theta$

Entering square $F_3\ B_1$

Exiting square F_2B_2

$\theta = 10$

	B_1	B_2	B_3
F_1	10	25	
F_2			25
F_3	10		5

d

$S(F_1) = 0 \quad S(F_2) = 0 \quad S(F_3) = -1$

$D(B_1) = 10 \quad D(B_2) = 4 \quad D(B_3) = 8$

$I_{13} = 11 - 0 - 8 = 3$

$I_{21} = 12 - 0 - 10 = 2$

$I_{22} = 5 - 0 - 4 = 1$

$I_{32} = 6 - (-1) - 4 = 3$

all positions are positive $\therefore$ optimal

$\text{Cost} = (10\times10) + (25\times4) + (25\times8) + (10\times9) + (5\times7) = 525 \text{ units}$

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Review Exercise 1
Exercise A, Question 31

Question:

A company makes three sizes of lamps, small, medium and large. The company is trying to determine how many of each size to make in a day, in order to maximise its profit. As part of the process the lamps need to be sanded, painted, dried and polished. A single machine carries out these tasks and is available 24 hours per day. A small lamp requires one hour on this machine, a medium lamp 2 hours and a large lamp 4 hours.

Let $x =$ number of small lamps made per day,
$y =$ number of medium lamps made per day,
$z =$ number of large lamps made per day,
where $x \geq 0, y \geq 0$ and $z \geq 0$.

a Write the information about this machine as a constraint.
b **i** Re-write your constraint from part **a** using a slack variable s.
ii Explain what s means in practical terms.

Another constraint and the objective function give the following simplex tableau. The profit P is stated in euros.

Basic variable	x	y	z	r	s	Value
r	3	5	6	1	0	50
s	1	2	4	0	1	24
P	−1	−3	−4	0	0	0

c Write down the profit on each small lamp.
d Use the simplex algorithm to solve this linear programming problem.
e Explain why the solution to part **d** is not practical.
f Find a practical solution which gives a profit of 30 euros. Verify that it is feasible. ***E***

Solution:

a $x+2y+4z \le 24$

b **i** $x+2y+4z+s=24$

ii $s(\ge 0)$ is the slack time on the machine in hours

c 1 euro

d

b.v.	x	y	z	r	s	value	
r	$\frac{3}{2}$	2	0	1	$-\frac{3}{2}$	14	$\text{R}1-6\text{R}2$
z	$\frac{1}{4}$	$\frac{1}{2}$	1	0	$\frac{1}{4}$	6	$\text{R}2 \div 4$
P	0	-1	0	0	1	24	$\text{R}3+4\text{R}2$

b.v.	x	y	z	t	s	value	
y	$\frac{3}{4}$	1	0	$\frac{1}{2}$	$\frac{-3}{4}$	7	$\text{R}1 \div 2$
z	$\frac{-1}{8}$	0	1	$\frac{-1}{4}$	$\frac{5}{8}$	$\frac{5}{2}$	$\text{R}2-\frac{1}{2}\text{R}1$
P	$\frac{3}{4}$	0	0	$\frac{1}{2}$	$\frac{1}{4}$	31	$\text{R}3+\text{R}1$

Profit = 31 euros $\quad y=7\ z=2.5\ x=r=s=0$

e Cannot make $\frac{1}{2}$ a lamp

f e.g. (0, 10, 0) or (0, 6, 3) or (1, 7, 2) checks in **both** inequalities

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Review Exercise 1
Exercise A, Question 32

Question:

	A	B	C	D	E	F	G	H
A	–	47	84	382	120	172	299	144
B	47	–	121	402	155	193	319	165
C	84	121	–	456	200	246	373	218
D	382	402	456	–	413	220	155	289
E	120	155	200	413	–	204	286	131
F	172	193	246	220	204	–	144	70
G	299	319	373	155	286	144	–	160
H	144	165	218	289	131	70	160	–

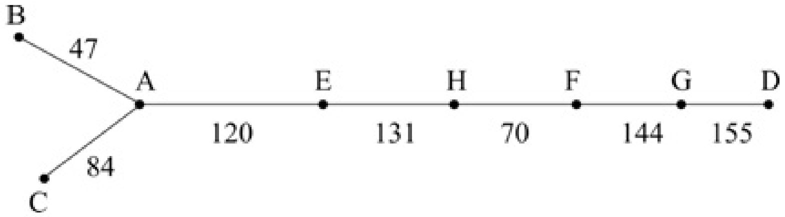
The table shows the distances, in miles, between some cities. A politician has to visit each city, starting and finishing at A. She wishes to minimise her total travelling distance.

a Find a minimum spanning tree for this network.
b Hence find an upper bound for this problem.
c Reduce this upper bound to a value below 1400 by using 'short cuts'.
d By deleting D find a lower bound for the distance to be travelled.
e Explain why the method used in part **d** will always give a lower bound for the distance to be travelled in any such network. ***E***

Solution:

a

	A	B	C	D	E	F	G	H
A	–	47	84	382	120	172	299	144
B	47	–	121	402	155	193	319	165
C	84	121	–	456	200	246	373	218
D	382	402	456	–	413	220	155	289
E	120	155	200	413	–	204	286	131
F	172	193	246	220	204	–	144	70
G	299	319	373	155	286	144	–	160
H	144	165	218	289	131	70	160	–



b Upper bound $751 \times 2 = 1502$

c B to D saves 265, H to G saves 54, B to H saves 133 etc.

d Delete D minimum spanning tree 596
2 least paths $155 + 220 = 375$
$\therefore$ lower bound is $596 + 375 = 971$

e The non-deleted vertices form a minimum spanning tree so they do not form a cycle.
The optimum solution is a cycle.
Unless the 2 least paths complete the cycle it will not give the optimum solution.
In general this will not be the case so a lower bound will be formed, shorter than the optimum solution:

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Review Exercise 1
Exercise A, Question 33

Question:

A carpenter makes small, medium and large chests of drawers. The small size requires $2\frac{1}{2}$ m of board, the medium size 10 m of board and the large size 15 m of board. The times required to produce a small chest, a medium chest and a large chest are 10 hours, 20 hours and 50 hours respectively. In a given year there are 300 m of board available and 1000 production hours available.

Let the number of small, medium and large chests made in the year be x, y and z respectively.

a Show that the above information leads to the inequalities
$x+4y+6z \leq 120,$
$x+2y+5z \leq 100.$
The profits made on small, medium and large chests are £10, £20 and £28 respectively.

b Write down an expression for the profit £P in terms of x, y and z.
The carpenter wishes to maximise his profit. The simplex algorithm is to be used to solve this problem.
c Write down the initial tableau using r and s as slack variables.
d Use two iterations of the simplex algorithm to obtain the following tableau. In the first iteration you should increase y.

Basic variable	x	y	z	r	s	Value
y	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	10
x	1	0	4	-1	2	80
P	0	0	22	0	10	1000

e Give a reason why this tableau is optimal.
f Write down the number of each type of chest that should be made to maximise the profit. State the maximum profit. ***E***

Solution:

a

	Board (m)	Time (R)
Small (x)	$2\frac{1}{2}$	10
Medium (y)	10	20
Large (z)	15	50
Available	300	1,000

Board $2\frac{1}{2}x+10y+15z \le 300$

$x+4y+6z \le 120$

Time $10x+20y+50z \le 1000$

$x+2y+5z \le 100$

b $P=10x+20y+28z$

c

b.v.	x	y	z	r	s	values
r	1	4	6	1	0	120
s	1	2	5	0	1	100
P	-10	-20	-28	0	0	0

d $\theta_1=30, \theta_2=50$; pivot 4

b.v.	x	y	z	r	s	value
y	$\frac{1}{4}$	1	$1\frac{1}{2}$	$\frac{1}{4}$	0	30
s	$\frac{1}{2}$	0	2	$\frac{-1}{2}$	1	40
P	-5	0	2	5	0	600

$\theta_1=120, \theta_2=80$; pivot $\frac{1}{2}$

b.v.	x	y	z	r	s	Value
y	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	10
x	1	0	4	-1	2	80
P	0	0	22	0	10	1000

e This tableau is optimal as there are no negative numbers in the profit line.

f Small 80, medium 10; large 0
Profit £1000

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Review Exercise 1
Exercise A, Question 34

Question:

	A	B	C	D	E	F	G
A	–	55	125	160	135	65	95
B	55	–	82	135	140	100	83
C	125	82	–	85	120	140	76
D	160	135	85	–	65	132	63
E	135	140	120	65	–	90	55
F	65	100	140	132	90	–	75
G	95	83	76	63	55	75	–

A retailer has shops in seven cities A, B, C, D, E, F and G. The table above shows the distances, in km, between each of these seven cities. Susie lives in city A and has to visit each of the shops. She wishes to plan a route starting and finishing at A and covering a minimum distance.

a Starting at A, use an algorithm to find a minimum spanning tree for this network. State the order in which you added vertices to the tree and draw your final tree. Explain briefly how you applied the algorithm.

b Hence determine an initial upper bound for the length of Susie's route.

c Starting from your initial upper bound, obtain an upper bound for the route which is less than 635 km. State the route which has a length equal to your new upper bound and cities which are visited more than once.

d Obtain the minimum spanning tree for the reduced graph produced by deleting the vertex G and all edges joined to it. Draw the tree.

e Hence obtain a lower bound for the length of Susie's route.

f Using your solution to part **d**, obtain a route of length less than 500 km which visits each vertex exactly once. ***E***

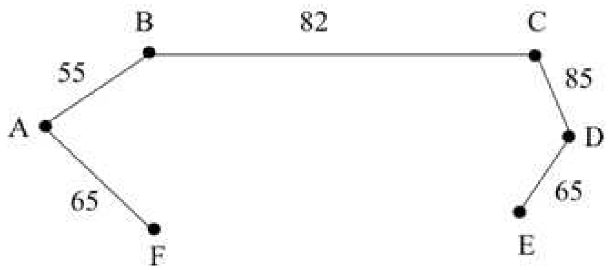
Solution:

a Label column A, delete row A.
Scan all labelled columns and choose least number.
Add that new vertex to the tree
Label the new vertex's column and delete its row
Repeat the 3 steps until all vertices added.
Applying algorithm
order of vertex selection A, B, F, G, E, D, C

b Initial upper band $= 2 \times 389 = 778$ km

c Reducing upper bound by short cuts
e.g. Using BC instead of $BA + AF + FG + GC$ leaves an upper bound of 589
Lists new route e.g. A B C G D G E G F A
States revisited vertices e.g. G

d



e Lower bound $= 352 + GD + GE$
$= 352 + 63 + 55$
$= 470$

f e.g. Use GE and GF (rather than GD)
length $= 352 + 55 + 75 = 482$ km
Route A B C D E G F A

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Review Exercise 1
Exercise A, Question 35

Question:

T42 Co. Ltd produces three different blends of tea, Morning, Afternoon and Evening. The teas must be processed, blended and then packed for distribution.
The table below shows the time taken, in hours, for each stage of the production of a tonne of tea. It also shows the profit, in hundreds of pounds, on each tonne.

	Processing	**Blending**	**Packing**	**Profit** (£100)
Morning blend	3	1	2	4
Afternoon blend	2	3	4	5
Evening blend	4	2	3	3

The total times available each week for processing, blending and packing are 35, 20 and 24 hours respectively. T42 Co. Ltd wishes to maximise the weekly profit.
Let x, y and z be the number of tonnes of Morning, Afternoon and Evening blend produced each week.

a Formulate the above situation as a linear programming problem, listing clearly the objective function, and the constraints as inequalities.

An initial simplex tableau for the above situation is

Basic variable	x	y	z	r	s	t	**Value**
r	3	2	4	1	0	0	35
s	1	3	2	0	1	0	20
t	2	4	3	0	0	1	24
P	−4	−5	−3	0	0	0	0

b Solve this linear programming problem using the simplex algorithm. Take the most negative number in the profit row to indicate the pivot column at each stage.

T42 Co. Ltd wishes to increase its profit further and is prepared to increase the time available for processing or blending or packing or any two of these three.

c Use your answer to part **b** to advise the company as to which stage(s) should be allocated increased time. ***E***

Solution:

a Objective: Maximise $P = 4x + 5y + 3z$

Subject to $3x + 2y + 4z \le 35$

$x + 3y + 2z \le 20$

$2x + 4y + 3z \le 24$

b

b.v.	x	y	z	r	s	t	Value	
r	2	0	$\frac{5}{4}$	1	0	$-\frac{1}{2}$	23	R1 − 2R3
s	$-\frac{1}{2}$	0	$-\frac{1}{4}$	0	1	$-\frac{3}{4}$	2	R2 − 3R3
y	$\frac{1}{2}$	1	$\frac{3}{4}$	0	0	$\frac{1}{4}$	6	R3 ÷ 4
P	$-\frac{3}{2}$	0	$\frac{3}{4}$	0	0	$\frac{5}{4}$	30	R4 + 5R3

b.v.	x	y	z	r	s	t	Value	
x	1	0	$\frac{5}{8}$	$\frac{1}{2}$	0	$-\frac{1}{4}$	$\frac{23}{2}$	R1 ÷ 2
s	0	0	$\frac{3}{8}$	$\frac{1}{4}$	1	$-\frac{7}{8}$	$\frac{31}{4}$	R2 + $\frac{1}{2}$R1
y	0	1	$\frac{1}{8}$	$-\frac{1}{4}$	0	$\frac{3}{8}$	$\frac{1}{4}$	R3 − $\frac{1}{2}$R1
P	0	0	$\frac{21}{8}$	$\frac{3}{4}$	0	$\frac{7}{8}$	$\frac{189}{4}$	R4 + $\frac{3}{2}$R1

$P = 47\frac{1}{4}$ $x = 11\frac{1}{2}, y = \frac{1}{4}, z = 0$

c There is some slack $(7\frac{3}{4})$ on S, so *do not* increase blending: therefore increase Processing and Packing which are both at their limit at present.

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Review Exercise 2
Exercise A, Question 1

Question:

In a game theory explain what is meant by

a zero-sum game,

b saddle point. ***E***

Solution:

a A game in which the gain to one player is equal to the loss of the other

b If there is a *stable solution*(s) a_{ij} in a game, the *location* of this stable solution is called the saddle point.
It is the point(s) where row maximin = column minimax

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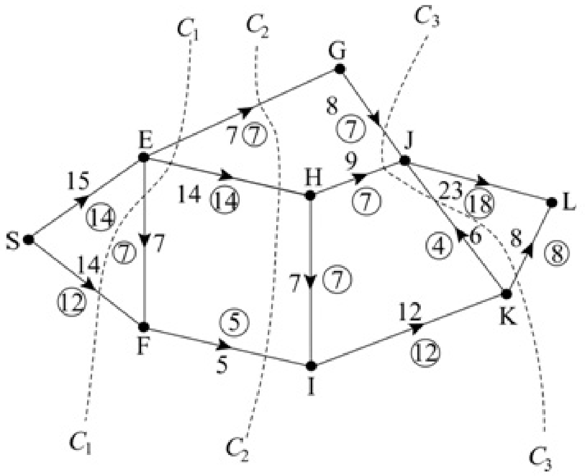
Review Exercise 2
Exercise A, Question 2

Question:



The diagram shows a network of roads represented by arcs. The capacity of the road represented by that arc is shown on each arc. The numbers in circles represent a possible flow of 26 from B to L.

Three cuts C_1, C_2 and C_3 are shown.

a Find the capacity of each of the three cuts.
b Verify that the flow of 26 is maximal.
The government aims to maximise the possible flow from S to L by using one of two options.
Option 1: Build a new road from E to J with capacity 5.
or
Option 2: Build a new road from F to H with capacity 3.
c By considering **both** options, explain which one meets the government's aim. ***E***

Solution:

a $C_1 = 7 + 14 + 0 + 14 = 35$
$C_2 = 7 + 14 + 5 = 26$
$C_3 = 8 + 9 + 6 + 8 = 31$

b Either Min cut = Max flow and we have a flow of 26 and a cut of 26
or C2 is through saturated arcs

c Using EJ (capacity 5) e.g. – will increase flow by 1 – i.e. increase it to 27 since only one more unit can leave E. – BEJL – 1
Using FH (capacity 3) e.g. – will increase flow by 2 – i.e. increase it to 28 since only two more units can leave F. – BFHJL – 2

Thus choose option 2 add FH capacity 3.

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Review Exercise 2
Exercise A, Question 3

Question:

A two person zero-sum game is represented by the following pay-off matrix for player A.

	B plays I	B plays II	B plays III
A plays I	−3	2	5
A plays II	4	−1	−4

a Write down the pay off matrix for player B.

b Formulate the game as a linear programming problem for player B, writing the constraints as equalities and stating your variables clearly. ***E***

Solution:

a

	A(I)	A(II)
B(I)	3	−4
B(II)	−2	1
B(III)	−5	4

b Add 6 to each element to make all terms positive

	A(I)	A(II)
B(I)	9	2
B(II)	4	7
B(III)	1	10

Let q_1 be the probability that B plays row 1
Let q_2 be the probability that B plays row 2
Let q_3 be the probability that B plays row 3
Let value of the game be v and let $V = v + 6$
where $q_1, q_2, q_3 \geq 0$
e.g. maximise $P = V$
Subject to $V - 9q_1 - 4q_2 - q_3 + r = 0$
$V - 2q_1 - 7q_2 - 10q_3 + s = 0$
$q_1 + q_2 + q_3 + t = 1$

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Review Exercise 2
Exercise A, Question 4

Question:

An engineering firm makes motors. They can make up to five in any one month, but if they make more than four they have to hire additional premises at a cost of £500 per month. They can store up to two motors for £100 per motor per month. The overhead costs are £200 in any month in which work is done.

Motors are delivered to buyers at the end of each month. There are no motors in stock at the beginning of May and there should be none in stock after the September delivery.

The order book for motors is:

Month	May	June	July	Aug.	Sept.
Number of motors	3	3	7	5	4

Use dynamic programming to determine the production schedule that minimises the costs, showing your working in a table. ***E***

Solution:

e.g.

Stage	State	Action	Dest	Value
1 (Sept)	2	2	0	$200+200=400*$
	1	3	0	$200+100=300*$
	0	4	0	$200=200*$
2 (Aug)	2	5	2	$200+200+500+400=1300$
		4	1	$200+200+300=700$
		3	0	$200+200+200=600*$
	1	5	1	$200+100+500+300=1100$
		4	0	$200+100+200=500*$
	0	5	0	$200+500+200=900*$
3 (July)	2	5	0	$200+200+500+900=1800*$
4 (June)	2	3	2	$200+200+1800=2200*$
	1	4	2	$200+100+1800=2100*$
	0	5	2	$200+500+1800=2500*$
5 (May)	0	5	2	$200+500+2200=2900$
		4	1	$200+2100=2300*$
		3	0	$200+2500=2700$

Month	May	June	July	August	September
Production schedule	4	4	5	5	4

Cost £2300

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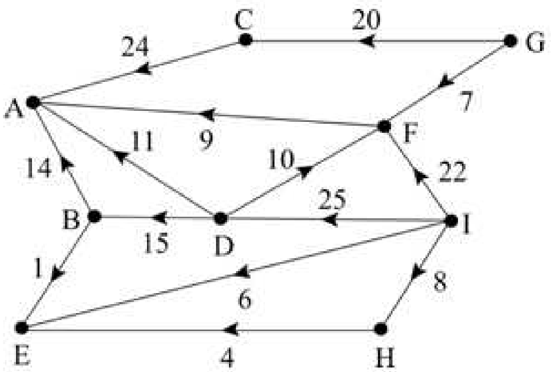
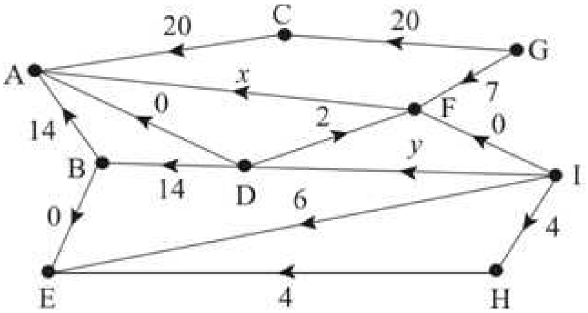
Review Exercise 2
Exercise A, Question 5

Question:



The diagram shows a capacitated directed network. The number on each arc is its capacity.



This shows a feasible initial flow through the same network.

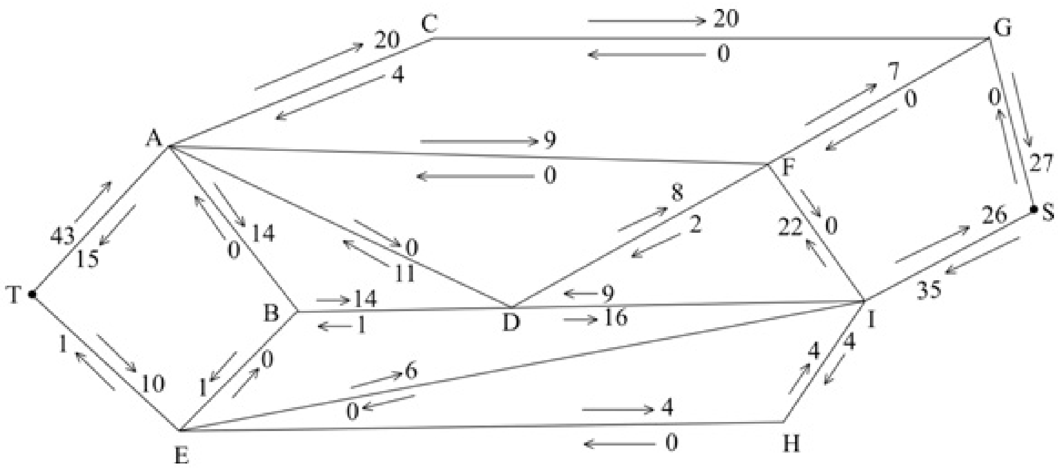
a Write down the values of the flow x and the flow y.
b Obtain the value of the initial flow through the network, and explain how you know it is not maximal.
c Use this initial flow and the labelling procedure to find a maximum flow through the network. You must list each flow-augmenting route you use, together with its flow.
d Show your maximal flow pattern.
e Prove that your flow is maximal. ***E***

Solution:

a $x = 9, y = 16$

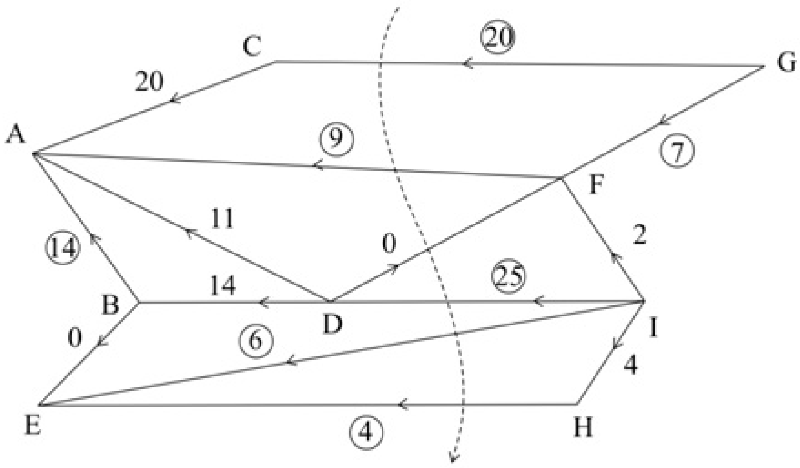
b Initial flow = 53 – either finds a flow-augmenting route or demonstrates not enough saturated arcs for a minimum cut

c



e.g. IDA – 9
IFDA – 24
max flow – 64

d



e Max flow – min cut
Finds a cut GC, AF, DF, DI, EI, EH value 64
Note: must not use supersource or supersink arcs.

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Review Exercise 2
Exercise A, Question 6

Question:

A two-person zero-sum game is represented by the following pay-off matrix for player A.

		B I	II	III	IV
	I	−4	−5	−2	4
A	**II**	−1	1	−1	2
	III	0	5	−2	−4
	IV	−1	3	−1	1

a Determine the play safe strategy for each player.
b Verify that there is a stable solution and determine the saddle points.
c State the value of the game to B. ***E***

Solution:

a Row minima $-5, -1, -4, -1$ max is -1
Column maxima $0, 5, -1, 4$ min is -1
Play safe is A plays II or IV and B plays III

b Since $(-1)-(-1) \equiv (-1)+1 = 0$ there is a stable solution
Saddle points (II, III) and (IV, III)

c Value of game to B is $-(-1) = 1$

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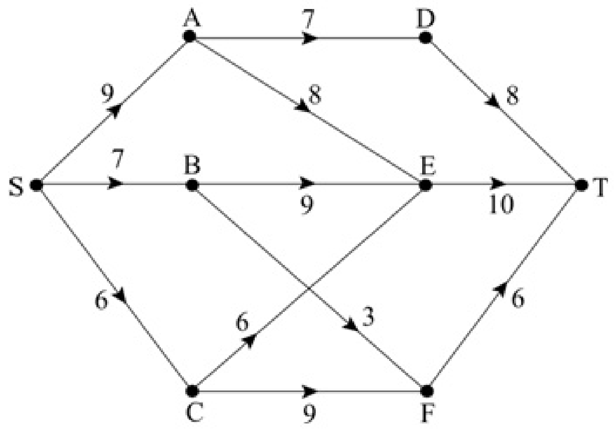
Review Exercise 2
Exercise A, Question 7

Question:



The network shows possible routes that an aircraft can take from S to T. The numbers on the directed arcs give the amount of fuel used on that part of the route, in appropriate units. The airline wishes to choose the route for which the maximum amount of fuel used on any part of the route is as small as possible. This is the minimax route.

a Complete a table to show the information.
b Hence obtain the minimax route from S to T and state the maximum amount of fuel used on any part of this route. ***E***

Solution:

a

Stage	Initial state	Action	Destination	Value
1	D	DT	T	8*
	E	ET	T	10*
	F	FT	T	6*
2	A	AD	D	$\max(7,8)=8$*
		AE	E	$\max(8,10)=10$
	B	BE	E	$\max(9,10)=10$
		BF	F	$\max(3,6)=6$*
	C	CE	E	$\max(6,10)=10$
		CF	F	$\max(9,6)=9$*
3		SA	A	$\max(9,8)=9$
	S	SB	B	$\max(7,6)=7$*
		SC	C	$\max(6,9)=9$

b Minimax route is S B F T
Maximum amount of fuel used is 7 units

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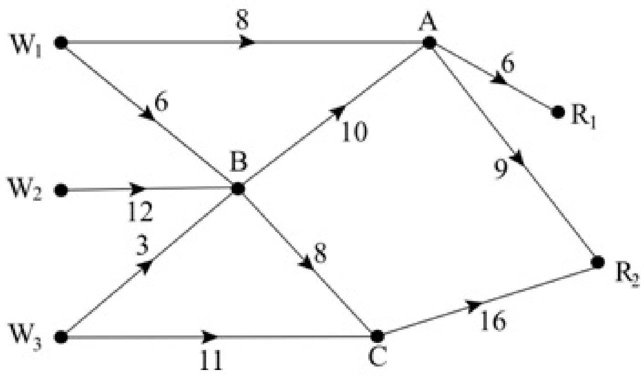
Review Exercise 2
Exercise A, Question 8

Question:



A company has three warehouses W_1, W_2 and W_3. It needs to transport the goods stored there to two retail outlets R_1 and R_2. The capacities of the possible routes, in van loads per day, are shown. Warehouses W_1, W_2 and W_3 have 14, 12 and 14 van loads respectively available per day and retail outlets R_1 and R_2 can accept 6 and 25 van loads respectively per day.

a On a copy of the diagram add a supersource W, a supersink R and the appropriate directed arcs to obtain a single-source, single-sink capacitated network. State the minimum capacity of each arc you have added.

b State the maximum flow along

i W W_1 A R_1 R ,

ii W W_3 C R_2 R .

c Taking your answers to part **b** as the initial flow pattern, use the labelling procedure to obtain a maximum flow through the network from W to R. Show your working. List each flow-augmenting route you use, together with its flow.

d From your final flow pattern, determine the number of van loads passing through B each day.

The company has the opportunity to increase the number of van loads from one of the warehouses W_1, W_2, W_3 to A, B or C.

e Determine how the company should use this opportunity so that it achieves a maximum flow. ***E***

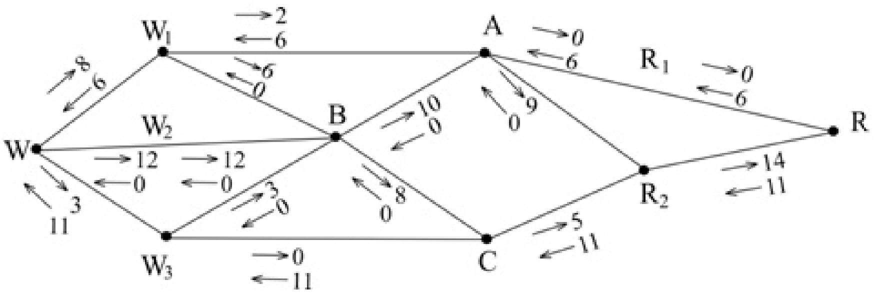
Solution:

a

b **i** $WW_1 A R_1 R - 6$

ii $WW_3 C R_2 R - 11$

c



e.g.

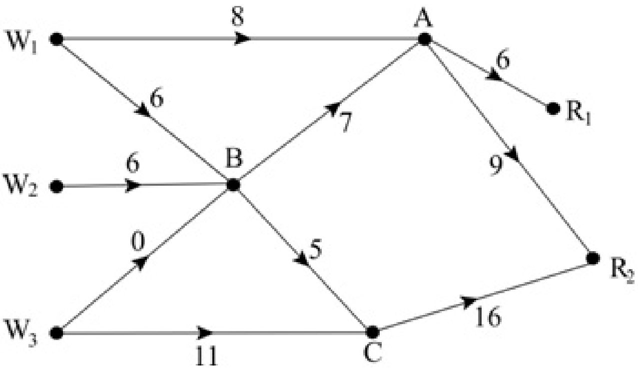
$W W_1 B A R_2 R - 6$

$W W_1 A R_2 R - 2$

$W W_2 B C R_2 R - 5$

$W W_2 B A R_2 R - 1$

Max flow 31



d 12 for this network (but may be different for other solutions)

e No use.
All arcs out of A and C are saturated, so the total flow cannot be increased unless the number of van loads from A or C to R_1 or R_2 is increased

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Review Exercise 2
Exercise A, Question 9

Question:

Emma and Freddie play a zero-sum game. This game is represented by the following pay-off matrix for Emma.

$$\begin{pmatrix} -4 & -1 & 3 \\ 2 & 1 & -2 \end{pmatrix}$$

a Show that there is no stable solution.

b Find the best strategy for Emma and the value of the game to her.

c Write down the value of the game to Freddie and his pay-off matrix. ***E***

Solution:

a

$$\begin{array}{cc} & \\ \text{Col max} & \begin{pmatrix} -4 & -1 & 3 \\ 2 & 1 & -2 \end{pmatrix} \\ & \begin{array}{ccc} 2 & 1 & 3 \end{array} \\ & \uparrow \\ & \text{min} \end{array} \quad \begin{array}{c} \text{row min} \\ -4 \\ -2 \end{array} \begin{array}{c} \\ \leftarrow \text{max} \\ \\ \end{array}$$

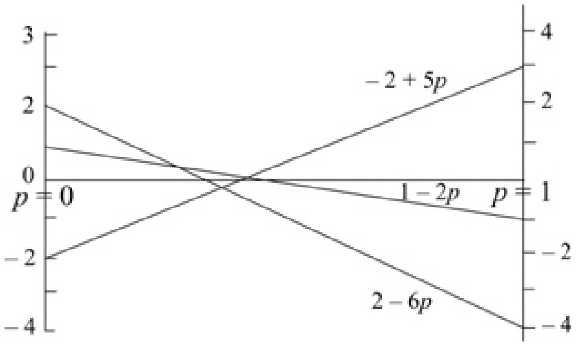
$-2 \neq 1 \quad \therefore$ not stable

b Let Emma play R_1 with probability p

If Freddie plays C_1 Emma's winnings are $-4p + 2(1-p) = 2-6p$

If Freddie plays C_2 Emma's winnings are $-p + 1(1-p) = 1-2p$

If Freddie plays C_3 Emma's winnings are $3p - 2(1-p) = -2+5p$



need intersection of $2-6p$ and $-2+5p$

$$2-6p = -2+5p$$
$$4 = 11p$$
$$p = \frac{4}{11}$$

So Emma should play R_1 with probability $\frac{4}{11}$

R_2 with probability $\frac{7}{11}$

The value of the game is $\frac{-2}{11}$ to Emma

c Value to Freddie $\frac{2}{11}$, matrix $\begin{pmatrix} 4 & -2 \\ 1 & -1 \\ -3 & 2 \end{pmatrix}$

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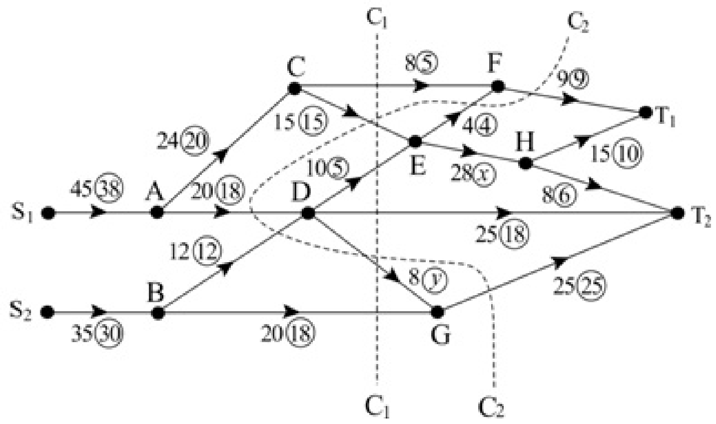
Review Exercise 2
Exercise A, Question 10

Question:



The diagram shows a capacitated, directed network. The unbracketed number on each arc indicates the capacity of that arc, and the numbers in circles show a feasible flow of value 68 through the network.

a Add a supersource and a supersink, and arcs of appropriate capacity, to a copy of the diagram.

b Find the values of x and y, explaining your method briefly.

c Find the value of cuts C_1 and C_2.

Starting with the given feasible flow of 68,

d use the labelling procedure to find a maximum flow through this network. List each flow-augmenting route you use, together with its flow.

e Show your maximum flow and state its value.

f Prove that your flow is maximal. ***E***

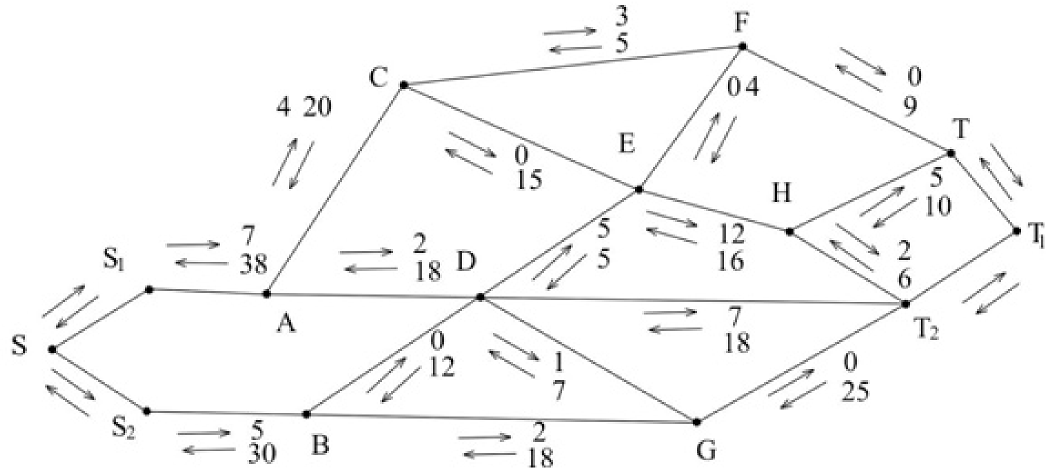
Solution:

a Adds S and T and arcs
$SS_1 \geq 45, SS_2 \geq 35, T_1T \geq 24, T_2T \geq 58$

b Using conservation of flow through vertices $x = 16$ and $y = 7$

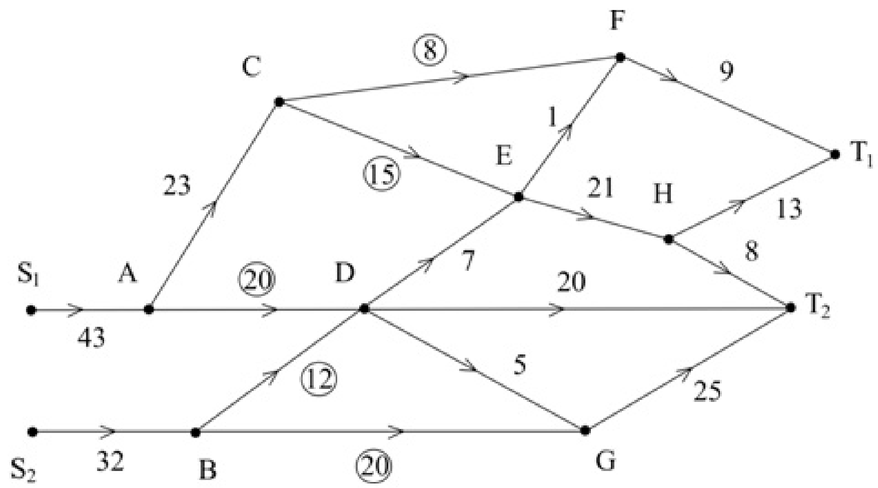
c $C_1 = 86, C_2 = 81$

d



SS_1 A D E H T_2 T − 2
e.g. SS_1 A C F E H T_1T − 3
SS_2 B G D T_2T − 2

e For example:



Flow 75

f Max flow – min cut theorem cut through CF, CE, AD, BD, BG (value 75)

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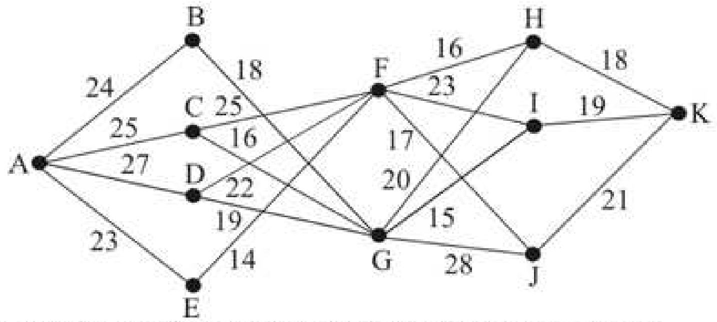
Review Exercise 2
Exercise A, Question 11

Question:

a Explain what is meant by a maximin route in dynamic programming, and give an example of a situation that would require a maximin solution.



A maximin route is to be found through the network shown.

b Complete the table on the worksheet, and hence find a maximin route.

c List **all** other maximin routes through the network. *E*

Solution:

a The route from start to finish in which the arc of minimum length is as large as possible.
Example must be practical, involve choice of route, have arc 'costs'.

b e.g. A company is planning its strategy for the next 4 years.
The number on each arc represents the expected profit resulting from each action.
The company wishes to ensure the minimum yearly profit is as large as possible.

Stage	State	Action	Value
1	H	HK	18*
	I	IK	19*
	J	JK	21*
2	F	FH	min(16,18) = 16
		FI	min(23,19) = 19*
		FJ	min(17,21) = 17
	G	GH	min(20,18) = 18
		GI	min(15,19) = 15
		GJ	min(28,21) = 21*
3	B	BG	min(18,21) = 18*
	C	CF	min(25,19) = 19*
		CG	min(16,21) = 16
	D	DF	min(22,19) = 19*
		DG	min(19,21) = 19*
	E	EF	min(14,19) = 14*
4	A	AB	min(24,18) = 18
		AC	min(25,19) = 19*
		AD	min(27,19) = 19*
		AE	min(23,14) = 14

c Routes: ACFIK, ADFIK, ADGJK

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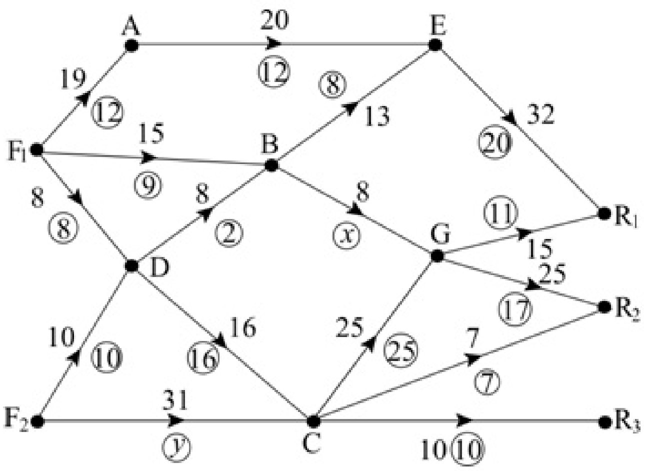
Review Exercise 2
Exercise A, Question 12

Question:



The diagram shows a capacitated, directed network of pipes flowing from two oil fields F_1 and F_2 to three refineries R_1, R_2 and R_3. The number on each arc represents the capacity of the pipe and the numbers in the circles represent a possible flow of 65.

a Find the value of x and the value of y.

b On the worksheet, add a supersource and a supersink, and arcs showing their minimum capacities.

c Taking the given flow of 65 as the initial flow pattern, use the labelling procedure to find the maximum flow. State clearly your flow augmenting routes.

d Show the maximum flow and write down its value.

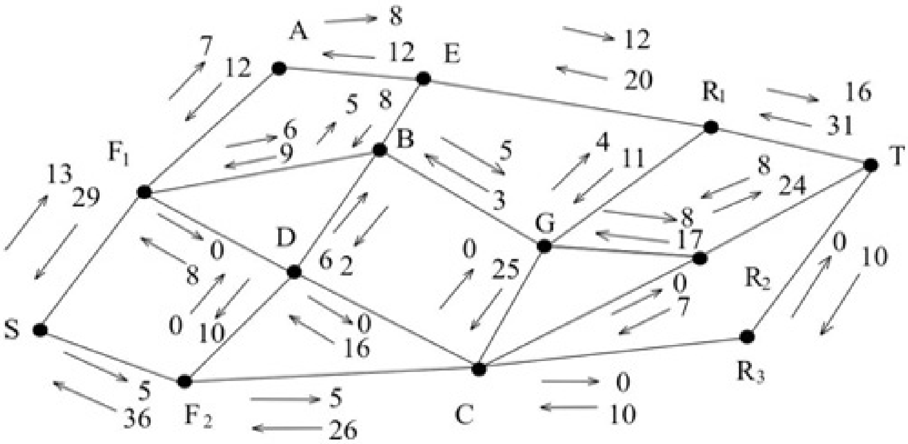
e Verify that this is the maximum flow by finding a cut equal to the flow. ***E***

Solution:

a $x = 3, y = 26$

b

c



e.g.

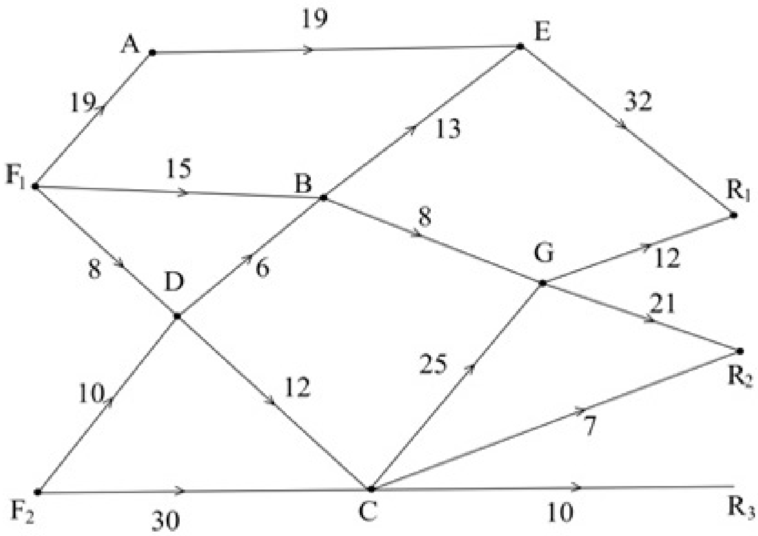
$SF_1AER_1T - 7$

$SF_1BER_1T - 5$

$SF_1BGR_1T - 1$

$SF_2CDBGR_2T - 4$

d e.g.



Max Flow 82

e e.g.

$F_1A, BE, BG, CG, CR_2, CR_3 (= 82)$

or $ER_1, BG, CG, CR_2, CR_3 (= 82)$

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Review Exercise 2
Exercise A, Question 13

Question:

A two person zero-sum game is represented by the following pay-off matrix for player A.

	B plays I	B plays II	B plays III
A plays I	2	−1	3
A plays II	1	3	0
A plays III	0	1	−3

a Identify the play safe strategies for each player.
b Verify that there is no stable solution to this game.
c Explain why the pay-off matrix above may be reduced to

	B plays I	B plays II	B plays III
A plays I	2	−1	3
A plays II	1	3	0

d Find the best strategy for player A, and the value of the game. ***E***

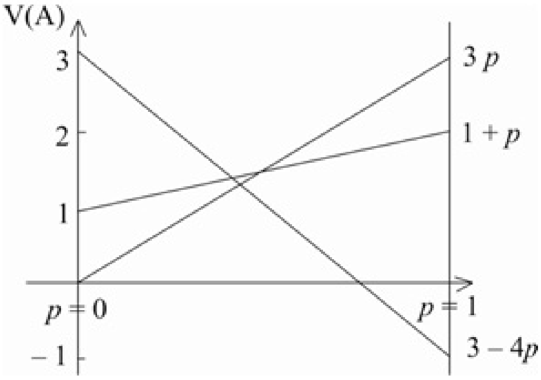
Solution:

a Player A: Row minima are $-1, 0, -3$ so maximin choice is play II
Player B: column maxima are 2, 3, 3 so minimax choice is play I

b Since A's maximin (0) $\neq$ B's minimax (2) no stable solution

c For player A row II dominates row III, (so A will never play III), since
$1 > 0 \quad 3 > 1 \quad 0 > -3$

d Let A play I with probability p and II with probability $(1-p)$
If B plays I A's expected winnings are $2p+(1-p)=1+p$
If B plays II A's expected winnings are $-p+3(1-p)=3-4p$
If B plays III A's expected winnings are $3p$



$3-4p=3p \Rightarrow p=\frac{3}{7}$

A should play I with probability $\frac{3}{7}$

II with probability $\frac{4}{7}$

and never play III

The value of the game is $\frac{9}{7}$ to A

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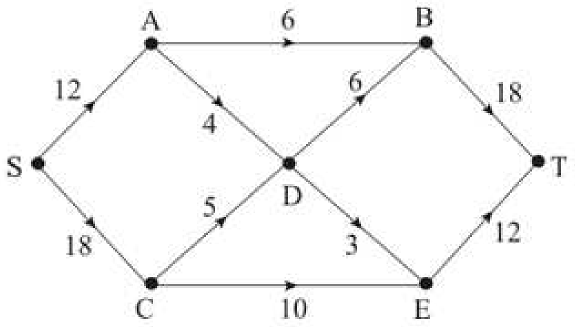
Review Exercise 2
Exercise A, Question 14

Question:



The diagram shows a capacitated network. The numbers on each arc indicate the capacity of that arc in appropriate units.

a Explain why it is not possible to achieve a flow of 30 through the network from S to T.
b State the maximum flow along
i SABT,
ii SCET.
c Show these flows on the worksheet.
d Taking your answer to part **c** as the initial flow pattern, use the labelling procedure to find a maximum flow from S to T. Show your working. List each flow-augmenting path you use together with its flow.
e Indicate a maximum flow.
f Prove that your flow is maximal. ***E***

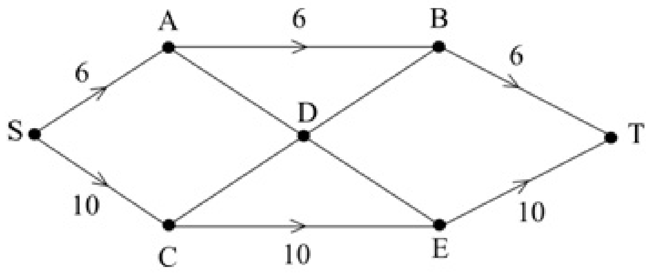
Solution:

a Finds a cut less than 30 giving its value.
e.g. cut through AB, AD, CD, CE (– 25) or AB, BD, ET (– 24)
or a consideration of flow input / flow output through A and C.

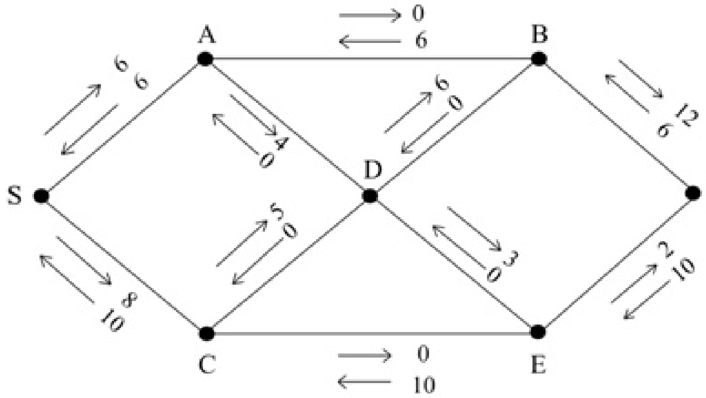
b **i** SABT (– 6)

ii SCET (10)

c

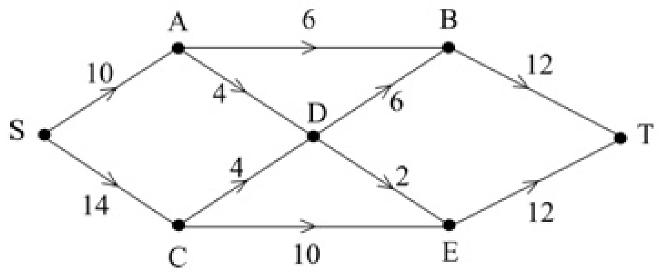


d



e.g. SADBT – 4
SCDBT – 2
SCDET – 2, max flow 24

e e.g.



f Refers to max flow – min cut theorem and the cut through AB, BD, ET of value 24.

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Review Exercise 2
Exercise A, Question 15

Question:

Andrew (A) and Barbara (B) play a zero-sum game. This game is represented by the following pay-off matrix for Andrew.

$$\begin{pmatrix} 3 & 5 & 4 \\ 1 & 4 & 2 \\ 6 & 3 & 7 \end{pmatrix}$$

a Explain why this matrix may be reduced to

$$\begin{pmatrix} 3 & 5 \\ 6 & 3 \end{pmatrix}.$$

b Hence find the best strategy for each player and the value of the game. ***E***

Solution:

a Row 1 dominates row 2 so A will never choose R2
Column 1 dominates column 3 so B will never choose C3
Thus Row 2 and column 3 may be deleted.

b Let A play row 1 with probability p and hence row 2 with probability $(1-p)$
If B plays 1 A's expected gain is $3p+6(1-p)=6-3p$
If B plays 2 A's expected gain is $5p+3(1-p)=2p+3$

Optimal when
$$\begin{aligned} 6-3p &= 2p+3 \\ 5p &= 3 \\ p &= \frac{3}{5} \end{aligned}$$

Hence A should play row 1 with probability $\frac{3}{5}$ and row 3 with probability $\frac{2}{5}$ and row 2 never

Similarly, let B play column 1 with probability q
$$3q+5(1-q)=6q+3(1-q) \Rightarrow 5-2q=3q+3$$
$$\begin{aligned} 5q &= 2 \\ q &= \frac{2}{5} \end{aligned}$$

So B should play column 1 with probability $\frac{2}{5}$ and column 2 with probability $\frac{3}{5}$ and column 3 never
Value of game is $4\frac{1}{5}$ to A

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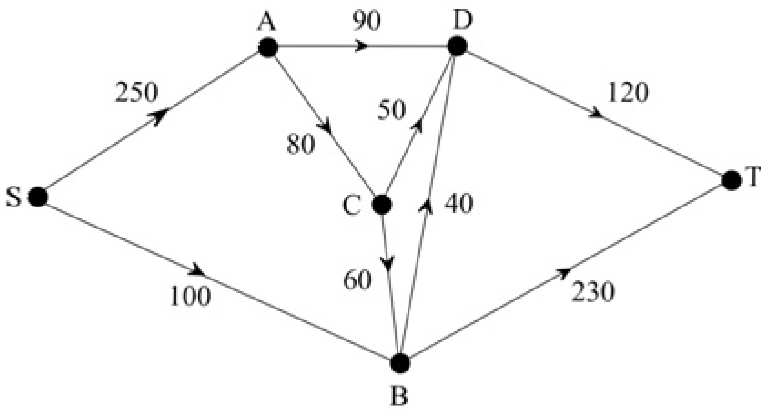
Review Exercise 2
Exercise A, Question 16

Question:



Natural gas is produced at S and is transported to a refinery at T by a network of underwater pipelines. The capacity of each pipeline, in appropriate units, is given in the diagram which shows the network of pipelines

a State the maximum flow along

i SACDT,

ii SBT.

b Show these two maximum flows on Diagram 1 of the worksheet.

c Taking your answer to part **b** as the initial flow pattern, use the labelling procedure to find a maximum flow from S to T showing your working on Diagram 2. List each flow augmenting route you find and state its flow.

d Show your maximum flow pattern on Diagram 3.

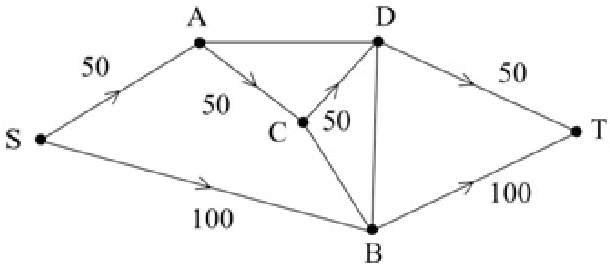
e Prove that your flow is maximal.

Solution:

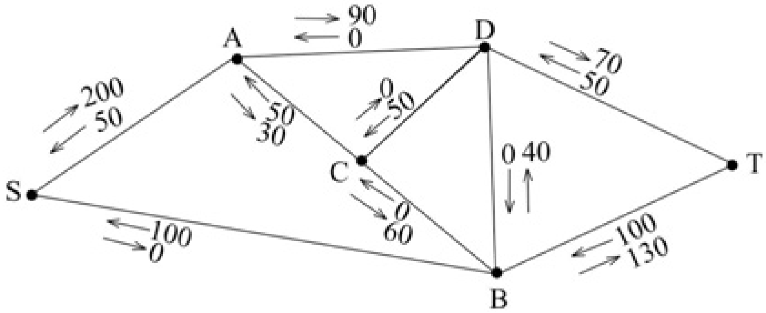
a **i** max flow along S A C D T = 50

ii max flow along S B T = 100

b



c



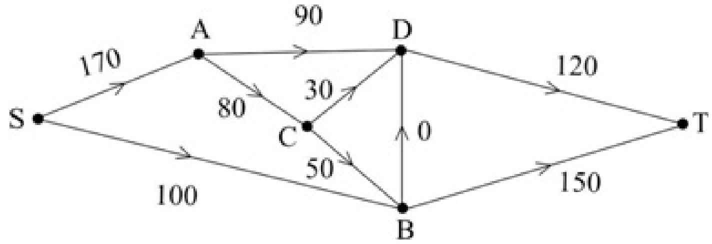
S A D T − 70

S A C B T − 30

S A D C B T − 20

Maximum flow 270

d



e Use max flow − min cut theorem

Cut though AD, AC and SB = 270 which equals flow

∴ maximal

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Review Exercise 2
Exercise A, Question 17

Question:

Kris produces custom made racing cycles. She can produce up to four cycles each month, but if she wishes to produce more than three in any one month she has to hire additional help at a cost of £350 for that month. In any month when cycles are produced, the overhead costs are £200. A maximum of three cycles can be held in stock in any one month, at a cost of £40 per cycle per month. Cycles must be delivered at the end of the month. The order book for cycles is

Month	Aug.	Sept.	Oct.	Nov.
Number of cycles required	3	3	5	2

Disregarding the cost of parts and Kris' time,

a determine the total cost of storing two cycles and producing four cycles in a given month, making your calculations clear.

There is no stock at the beginning of August and Kris plans to have no stock after the November delivery.

b Use dynamic programming to determine the production schedule which minimises the costs, showing your working in a table.

The fixed cost of parts is £600 per cycle and of Kris' time is £500 per month. She sells the cycles for £2000 each.

c Determine her total profit for the four-month period. ***E***

Solution:

a total cost $= 2\times40+350+200=£630$

b

Stage	Demand	State	Action	Destination	Value
(2) Oct	(5)	(1)	(4)	(0)	$(590+200=790)$ *
		(2)	(3)	(0)	$280+200=480$ *
			(4)	(1)	$630+240=870$
		(3)	(2)	0	$320+200=520$ *
			3	1	$320+240=560$
			4	2	$670+80=750$
3 Sept	3	0	4	1	$550+790=1340$ *
		1	3	1	$240+790=1030$ *
			4	2	$590+480=1070$
4 Aug	3	0	3	0	$200+1340=1540$ *
			4	1	$550+1030=1580$

Month	August	September	October	November
Make	3	4	4	2

cost $=£1540$

c Profit per cycle $=13\times1400$ cost of Kris' time $=£2000$

$=18\,200$ cost of production $=£1540$

$\therefore$ total profit $=18\,200-3540$

$=£14\,660$

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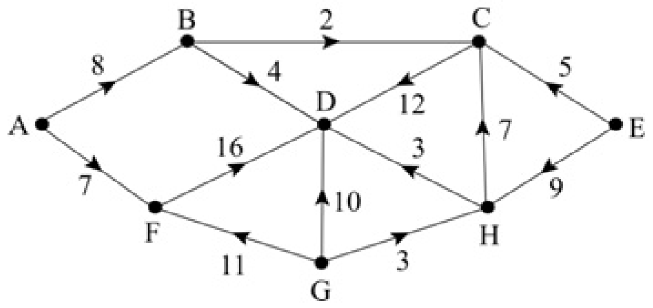
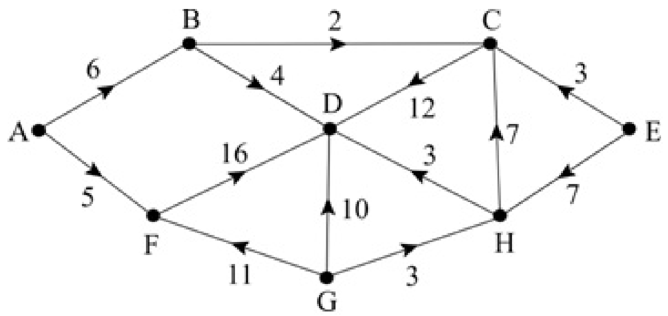
Review Exercise 2
Exercise A, Question 18

Question:



The network above models a drainage system. The number on each arc indicates the capacity of that arc, in litres per second.

a Write down the source vertices.



This network shows a feasible flow through the same network.

b State the value of the feasible flow shown.

Taking the flow shown as your initial flow pattern,

c use the labelling procedure to find a maximum flow through this network. You should list each flow-augmenting route you use, together with its flow.

d Show the maximum flow and state its value.

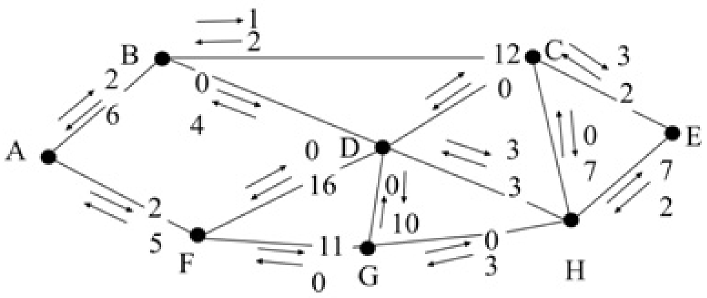
e Prove that your flow is maximal. ***E***

Solution:

a A. E and G

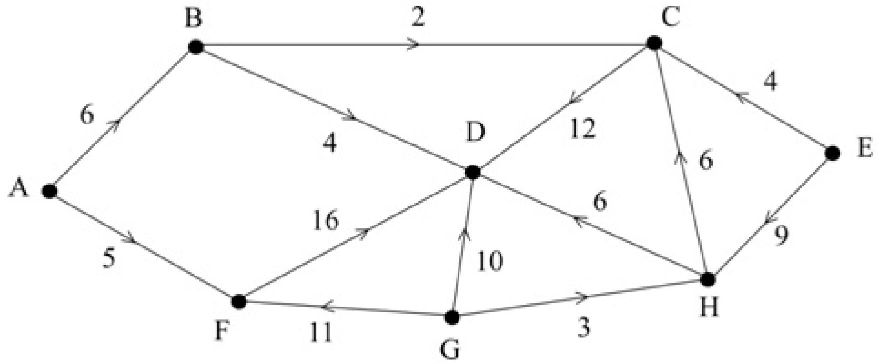
b 45

c



e.g $\begin{array}{l}\text{EHD} - 2\\ \text{ECHD} - 1\end{array}$

d



Maximum flow 48

e Max flow – Min cut theorem

Cut through DB, DC, DH, DG, DF

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Review Exercise 2
Exercise A, Question 19

Question:

A two-person zero-sum game is represented by the following pay-off matrix for player A.

	B plays 1	B plays 2	B plays 3	B plays 4
A plays 1	−2	1	3	−1
A plays 2	−1	3	2	1
A plays 3	−4	2	0	−1
A plays 4	1	−2	−1	3

a Verify that there is no stable solution to this game.
b Explain why the 4×4 game above may be reduced to the following 3×3 game.

−2	1	3
−1	3	2
1	−2	−1

c Formulate the 3×3 game as a linear programming problem for player A. Write the constraints as inequalities. Define your variables clearly. ***E***

Solution:

a Row minimums $\{-2, -1, -4, -2\}$ row maximin $= -1$
Column maximums $\{1, 3, 3, 3\}$ column minimax $= 1$
Since $1 \neq -1$ not stable

b Row 2 dominates Row 3
column 1 dominates column 4

c Let A play row R, with probability P_1, R_2 with probability P_2 and "R_3" with probability P_3

$$\begin{pmatrix} -2 & 1 & 3 \\ -1 & 3 & 2 \\ 1 & -2 & -1 \end{pmatrix} \underset{+3}{\xrightarrow{\text{e.g.}}} \begin{pmatrix} 1 & 4 & 6 \\ 2 & 6 & 5 \\ 4 & 1 & 2 \end{pmatrix}$$

e.g. maximise $P = V$

subject to
$$\begin{aligned} V - P_1 - 2P_2 - 4P_3 &\le 0 \\ V - 4P_1 - 6P_2 - P_3 &\le 0 \\ V - 6P_1 - 5P_2 - 2P_3 &\le 0 \\ P_1 + P_2 + P_3 &\le 1 \\ V_1\ P_1\ P_2\ P_3 &\ge 0 \end{aligned}$$

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Review Exercise 2
Exercise A, Question 20

Question:

a Explain briefly what is meant by a zero-sum game.
A two person zero-sum game is represented by the following pay-off matrix for player A.

	I	II	III
I	5	2	3
II	3	5	4

b Verify that there is no stable solution to this game.
c Find the best strategy for player A and the value of the game to her.
d Formulate the game as a linear programming problem for player B. Write the constraints as inequalities and define your variables clearly. ***E***

Solution:

a A zero-sum game is one in which the sum of the gains for all players is zero.

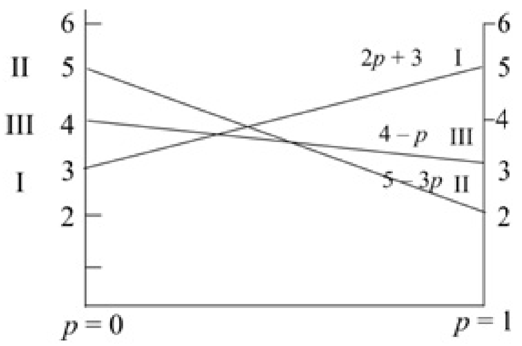
b

	I	II	III	
I	5	2	3	min 2
II	3	5	4	min 3 $\leftarrow$ max
	max 5	5	4	
			$\uparrow$	
			min	

Since $3 \neq 4$ not stable

c Let A play I with probability p
Let A play II with probability $(1-p)$

If B play I A's gains are $5p+3(1-p)=2p+3$
If B plays II A's gains are $2p+5(1-p)=5-3p$
If B plays III A's gains are $3p+4(1-p)=4-p$



Intersection of $2p+3$ and $4-p \Rightarrow p=\frac{1}{3}$

$\therefore$ A should play I $\frac{1}{3}$ of time and II $\frac{2}{3}$ of time; value (to A) $=3\frac{2}{3}$

d Let B play I with probability q_1, II with probability q_2 and III with probability q_3

e.g. $\begin{bmatrix} -5 & -3 \\ -2 & -5 \\ -3 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 3 & 2 \end{bmatrix}$

maximise $P=V$

$V-q_1-4q_2-3q_3 \leq 0$

Subject to $V-3q_1-q_2-2q_3 \leq 0 \quad q_1+q_2+q_3 \leq 1$

$v_1, q_1, q_2, q_3 \geq 0 \quad$ or $=1$

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Review Exercise 2
Exercise A, Question 21

Question:

Joan sells ice cream. She needs to decide which three shows to visit over a three-week period in the summer. She starts the three-week period at home and finishes at home. She will spend one week at each of the three shows she chooses, travelling directly from one show to the next.

Table 1 gives the week in which each show is held. Table 2 gives the expected profits from visiting each show. Table 3 gives the cost of travel between shows.

Table 1

Week	1	2	3
Shows	A, B, C	D, E	F, G, H

Table 2

Show	A	B	C	D
Expected profit (£)	900	800	1000	1500
Show	E	F	G	H
Expected profit (£)	1300	500	700	600

Table 3

Travel costs (£)	A	B	C	D	E	F	G	H
Home	70	80	150			80	90	70
A				180	150			
B				140	120			
C				200	210			
D						200	160	120
E						170	100	110

It is decided to use dynamic programming to find a schedule that maximises the total expected profit, taking into account the travel costs.

a Define suitable stage, state and action variables.

b Determine the schedule that maximises the total profit. Show your working in a table.

c Advise Joan on the shows that she should visit and state her total expected profit. ***E***

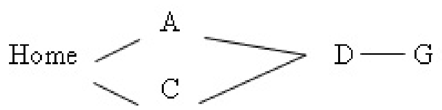
Solution:

a Stage – Number of weeks to finish
State – Show being attended
Action – Next journey to undertake

b

Stage	State	Action	Value
1	F	F – Home	$500 - 80 = 420*$
	G	G – Home	$700 - 90 = 610*$
	H	H – Home	$600 - 70 = 530*$
2	D	DF	$1500 - 200 + 420 = 1720$
		DG	$1500 - 160 + 610 = 1950*$
		DH	$1500 - 120 + 530 = 1910$
	E	EF	$1300 - 170 + 420 = 1550$
		EG	$1300 - 100 + 610 = 1810*$
		EH	$1300 - 110 + 530 = 1720$
3	A	AD	$900 - 180 + 1950 = 2670*$
		AE	$900 - 150 + 1810 = 2560$
	B	BD	$800 - 140 + 1950 = 2610*$
		BE	$800 - 120 + 1810 = 2490$
	C	CD	$1000 - 200 + 1950 = 2750*$
		CE	$1000 - 210 + 1810 = 2600$
4	Home	Home – A	$-70 + 2670 = 2600*$
		Home – B	$-80 + 2610 = 2530$
		Home – C	$-150 + 2750 = 2600*$

c



Total profit £2600